On determinants and eigenvalues

- Review: eigenvalues and eigenvectors.
- Eigenspaces.
- Characteristic equation.
- Multiplicity of eigenvalues.

The eigenvectors of a matrix determine directions where the action of the matrix is simple

$A$ is an $n \times n$ matrix. Then $\lambda$ is an eigenvalue of $A$ with eigenvector $x \neq 0 \iff Ax = \lambda x$.

Simple means $Ax$ is proportional to $x$.

A matrix may or may not have eigenvalues and eigenvectors.

$A$ is invertible $\iff \lambda = 0$ is not an eigenvalue of $A$. 

Examples of eigenvalues and eigenvectors

- The matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ has eigenvectors and eigenvalues
  $u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda_1 = 4$, $u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\lambda_2 = -2$.

- The matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has eigenvectors and eigenvalues
  $u_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\lambda_1 = 0$, $u_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\lambda_2 = 7$.

Eigenspaces are the subspaces spanned by the eigenvectors

**Definition 1** Let $\lambda$ be an eigenvalue of $A$. The eigenspace $E_A(\lambda)$ is the set of all vectors $x$ solutions of $Ax = \lambda x$.

**Theorem 1** If $\lambda$ is an eigenvector of an $n \times n$ matrix $A$, then the set $E_A(\lambda) \subseteq \mathbb{R}^n$ is a subspace.

Eigenspaces are lines, planes, or hyperplanes through the origin.
Here are the eigenspaces of the previous examples

The matrix \( A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \) has eigenvectors \( \lambda_1 = 4 \) and \( \lambda_2 = -2 \). The corresponding eigenspaces are

\[ E(4) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ t \in \mathbb{R} \right\}, \]

\[ E(-2) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x = t \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ t \in \mathbb{R} \right\}. \]

Here are the eigenspaces of the previous examples

The matrix \( A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \) has eigenvectors \( \lambda_1 = 0 \) and \( \lambda_2 = 7 \). The corresponding eigenspaces are

\[ E(0) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \ t \in \mathbb{R} \right\}, \]

\[ E(7) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x = t \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \ t \in \mathbb{R} \right\}. \]
Solving the characteristic equation one finds first the \( \lambda \)'s.

**Theorem 2** \( \lambda \) is an eigenvalue of \( A \) if and only if \( \det(A - \lambda I) = 0 \).

**Definition 2** Given an \( n \times n \) matrix \( A \), the function \( f(\lambda) = \det(A - \lambda I) \) is called the characteristic function of \( A \).

The characteristic function of \( A, n \times n \) is a polynomial in \( \lambda \) of degree \( n \).

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The multiplicity of eigenvalues is a way to count for repeated eigenvalues.

**Definition 3** Let \( f(\lambda) \) be the characteristic polynomial of an \( n \times n \) matrix. The eigenvalue \( \lambda_0 \) has algebraic multiplicity \( r > 0 \) if

\[
f(\lambda) = (\lambda - \lambda_0)^r g(\lambda), \quad \text{with} \quad g(\lambda_0) \neq 0.
\]
Eigenvalue with multiplicity 2

The eigenvalues of \( A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \) are given by

\[
\begin{vmatrix} 2 - \lambda & 3 \\ 0 & 2 - \lambda \end{vmatrix} = (2 - \lambda)^2 = 0,
\]

that is, \( \lambda = 2 \), which has multiplicity 2.

More examples of eigenvalues various multiplicities

Find the eigenvalues and eigenspaces of the following matrices:

\[
A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}.
\]

The both matrices have the same eigenvalues, because,

\[
f_A(\lambda) = f_B(\lambda) = (\lambda - 3)^2(1 - \lambda)
\]

so the eigenvalues are:

\( \lambda = 3 \) with multiplicity 2; and \( \lambda = 1 \) with multiplicity 1.
Once the eigenvalues are known, the eigenvectors can be easily computed.

If the eigenvalues $\lambda$ are known, then the eigenvector $x_\lambda$ is a solution of the homogeneous equation

$$(A - \lambda I)x_\lambda = 0.$$ 

Here are the eigenvectors of the previous example.

The eigenspaces of the matrix $B = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ are

$E_B(3) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, $E_B(1) = \left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \right\}$.

$\dim E_B(\lambda) = \text{mul}(\lambda)$ for every eigenvalue of $B$, the set of all eigenvectors of $B$ is a basis of $\mathbb{R}^3$. 

Here are the eigenvectors of the previous example

The eigenspaces of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ are

$$E_A(3) = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \} \quad E_A(1) = \{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \} \}.$$  

In the case of $A$, where for $\lambda = 3$ holds that $\dim E_A(3) < \text{multipl.}(3)$, the set of eigenvectors of $A$ is not a basis of $\mathbb{R}^3$.

In general $\dim E(\lambda) \leq \text{multipl.}(\lambda)$.