The beginning of Linear Algebra • Review - Broad overview. Slide 1 – Main results (so far). - Examples. • The need of abstraction: Vector Space.

Main results so far including an invertible matrix **Theorem 1** Let $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ be an $n \times n$ matrix. A is invertible • $\Leftrightarrow \exists A^{-1}, n \times n, such that (A^{-1})A = I = A(A^{-1});$ • $\Leftrightarrow A\mathbf{x} = \mathbf{b}$ has a unique solution \mathbf{x} for all $\mathbf{b} \in \mathbb{R}^n$. • \Leftrightarrow $Col(A) = \mathbb{R}^n;$ • $\Leftrightarrow N(A) = \{\mathbf{0}\};$ • $\Leftrightarrow \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ are l.i.;

Main results so far including an $m \times n$ matrix

Theorem 2 Let $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ be an $m \times n$ matrix.

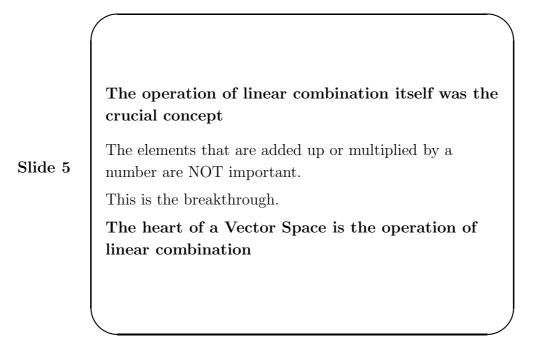
- $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} \in \mathbb{R}^n \Leftrightarrow \mathbf{b} \in Col(A) \subset \mathbb{R}^m$.
- The solution $\mathbf{x} \in \mathbb{R}^n$ above is unique $\Leftrightarrow N(A) = \{\mathbf{0}\}.$

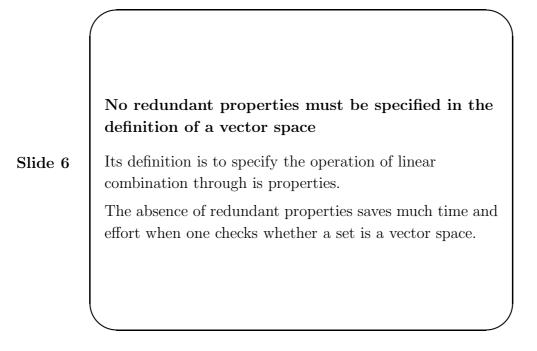
All these interrelated concepts must be organized in a unified subject

It was realized that the central concept unifying this subject was the one of *linear combination*:

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- column vectors: $\mathbf{u} = a\mathbf{v} + \mathbf{w};$
- $m \times n$ matrices: C = aA + bB;
- continuous functions: h(x) = af(x) + g(x);
- row vectors: $\mathbf{u}^T = a\mathbf{v}^T + b\mathbf{w}^T$.





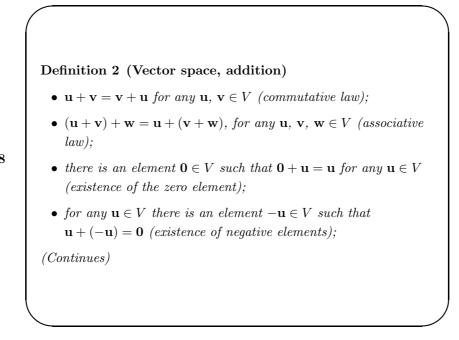
A vector space is defined by characterizing the operation linear combination

Definition 1 (Vector space) Let V be a set of objects, to be called vectors; and let \mathbb{R} be the real numbers. Assume that there are two operations;

 $(\mathbf{u}, \mathbf{v}) \to \mathbf{u} + \mathbf{v} \in V, \quad (a, \mathbf{v}) \to a\mathbf{v} \in V,$

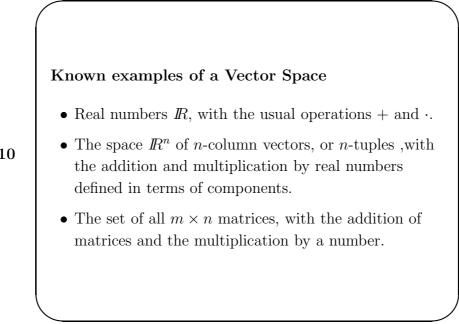
called addition and scalar multiplication, respectively, defined for any $\mathbf{u}, \mathbf{v} \in V$ and $a \in \mathbb{R}$. These operations are to satisfy the following rules.

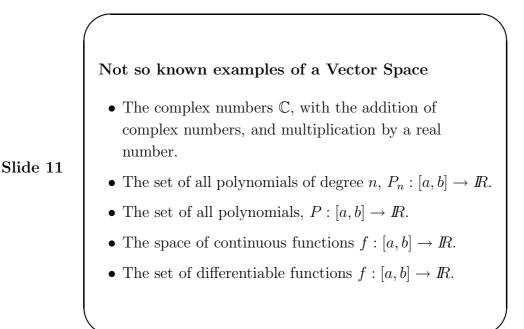
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Definition 3 (Vector space, multiplication by a number)
1u = u for any u ∈ V;
a(bu) = (ab)u for any u ∈ V, a, b ∈ ℝ (associative law for scalar multiplication);
a(u + v) = au + av and (a + b)u = au + bu for any u, v ∈ V, and a, b ∈ ℝ (distributive laws).
Then V is called a real vector space, or a linear space.





From the commutative law and the associative law we observe that to add several elements, the order of the summation does not matter, and it does not cause any ambiguity to write expression such $\mathbf{u} + \mathbf{v} + \mathbf{w}$ or $\sum_{i=1}^{n} \mathbf{u}_{1}$.

By using the commutative law and the associative law it is not difficult to verify that the zero element $\mathbf{0}$ and the negative element $-\mathbf{u}$ of a given element $\mathbf{u} \in V$ are unique, and they can be equivalently defined by the relations $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for any $\mathbf{u} \in V$, and $(-\mathbf{u}) + \mathbf{u} = \mathbf{0}$. Below we write $\mathbf{u} - \mathbf{v}$ for $\mathbf{u} + (\mathbf{v})$.

The zero element is unique

Proof: Suppose there are two elements, $\mathbf{0}_1$, $\mathbf{0}_2$ such that

$$\begin{aligned} \mathbf{0}_1 + \mathbf{u} &= \mathbf{u}, \quad \forall \mathbf{u} \in V, \\ \mathbf{0}_2 + \mathbf{u} &= \mathbf{u}, \quad \forall \mathbf{u} \in V, \end{aligned}$$

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then take $\mathbf{u} = \mathbf{0}_2$ in the first equation, and $\mathbf{u} = \mathbf{0}_1$ in the second equation, then one gets, respectively,

$$0_1 + 0_2 = 0_2, \quad 0_2 + 0_1 = 0_1.$$

The left hand sides are equal, because of the commutative law, the the right hand sides are equal, so $\mathbf{0}_1 = \mathbf{0}_2$.

This property is also deduced from the definition of vector space

Theorem 3 $0\mathbf{u} = \mathbf{0}$.

Proof: $\mathbf{u} = 1\mathbf{u} = (0+1)\mathbf{u} = 0\mathbf{u} + 1\mathbf{u} = 0\mathbf{u} + \mathbf{u}$, then

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 $0\mathbf{u} + \mathbf{u} = \mathbf{u},$

that is, $0\mathbf{u}$ is a zero element. But the zero element is unique, so

$$0\mathbf{u} = \mathbf{0}$$

 $\begin{cases} \text{The negative element is unique} \\ \text{Proof: Assume that there are two negative elements, } \mathbf{v}_1 \\ \text{and } \mathbf{v}_2 \text{ for } \mathbf{u}, \text{ that is,} \\ \mathbf{u} + \mathbf{v}_1 = \mathbf{0}, \quad \mathbf{u} + \mathbf{v}_2 = \mathbf{0}. \\ \text{Then, one has that} \\ \mathbf{v}_1 = \mathbf{0} + \mathbf{v}_1, \\ = \mathbf{u} + \mathbf{v}_2 + \mathbf{v}_1, \\ = \mathbf{u} + \mathbf{v}_1 + \mathbf{v}_2, \\ = \mathbf{0} + \mathbf{v}_2, \\ = \mathbf{v}_2. \end{cases}$

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Even this property comes from the definition Theorem 4 $(-\mathbf{u}) = (-1)\mathbf{u}$.

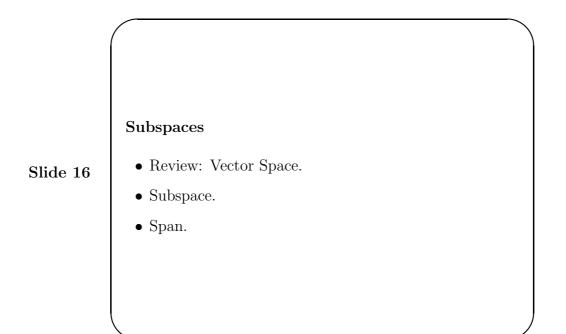
Proof:

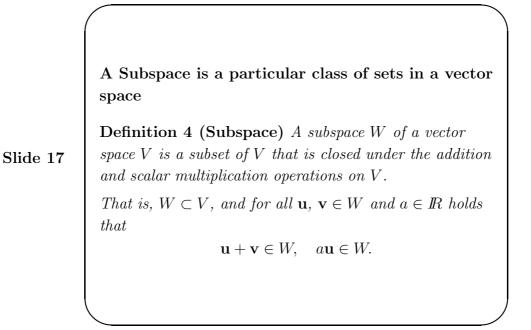
$$\begin{array}{rcl}
 0 &=& 0\mathbf{u}, \\
 &=& (1-1)\mathbf{u}, \\
 &=& 1\mathbf{u} + (-1)\mathbf{u}, \\
 &=& \mathbf{u} + (-1)\mathbf{u}, \\
 \end{array}$$

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therefore, $(-1)\mathbf{u}$ is a negative element of \mathbf{u} . But negative elements are unique, so,

$$(-1)\mathbf{u} = -\mathbf{u}.$$





Examples

• The set $W \subset \mathbb{R}^3$ given by $W = \{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = (x_1, x_2, 0), \quad x_1, x_2 \in \mathbb{R} \},$

is a subspace of \mathbb{R}^3 .

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• The set $\widehat{W} \subset \mathbb{R}^3$ given by $\widehat{W} = \{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = (x_1, x_2, 1), \quad x_1, x_2 \in \mathbb{R} \},\$

in contrast is not a subspace of \mathbb{R}^3 .

Span

Definition 5 The Span of $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$ is defined as the set of all linear combinations of these vectors, that is

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 $Span(\mathbf{v}_1,\cdots,\mathbf{v}_n)=\{\mathbf{x}=a_1\mathbf{v}_1+\cdots+a_n\mathbf{v}_n\},\$

where $a_1, \cdots, a_n \in \mathbb{R}$.

Theorem 5 Span $(\mathbf{v}_1, \cdots, \mathbf{v}_n) \subset V$ is a subspace of V.

Slide 20 $\widehat{W} = \operatorname{span}\{\mathbf{e}_{1}, \mathbf{e}_{2}\}, \\ = \operatorname{span}\{(1, 1, 0), (-1, 1, 0)\}, \\ = \operatorname{span}\{(2, 0, 0), (1, 1, 0), (12, -1, 0)\}, \\ = \operatorname{span}\{(2, 0, 0), (1, 1, 0), (12, -1, 0), (-3, -1, 0)\}.$

Examples
The set W = {(x₁, x₂) ∈ ℝ² : x₁ ≥ 0} is not a subspace of ℝ².
The segment W{x ∈ ℝ : −1 ≤ x ≤ 1} is not a subspace of ℝ.
The line W = {x ∈ ℝ³ : x = (1, 2, 3)t} is a subspace or ℝ³.
The line W = {x ∈ ℝ³ : x = (1, 2, 0) + (1, 2, 3)t} is not a subspace or ℝ³.