Print Name: ______________________ **Student Number:** _____________

Section Time: _____________

Math 20F.
Midterm Exam 1
February 8, 2006

Read each question carefully, and answer each question completely.
Show all of your work. No credit will be given for unsupported answers.
Write your solutions clearly and legibly. No credit will be given for illegible solutions.

1. (10 points) Consider the matrix A and the vector $\mathbf{b}$ given by

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -6 & 2 & 3 \\ -2 & 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

(a) Find the inverse of A. Justify your answer.

(b) Find the solution of the non-homogeneous system $A\mathbf{x} = \mathbf{b}$. Justify your answer.

(a)

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ -6 & 2 & 3 & 0 & 1 & 0 \\ -2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & -10 & -3 & 6 & 1 & 0 \\ 0 & -3 & -1 & 2 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3/10 & -3/5 & -1/10 & 0 \\ 0 & -3 & -1 & 2 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 0 & -2/5 & -1/5 & -1/5 & 0 \\ 0 & 1 & 3/10 & -3/5 & -1/10 & 0 \\ 0 & 0 & -1/10 & 1/5 & -3/10 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & -4 \\ 0 & 1 & 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & -2 & 3 & -10 \end{array}\right], \quad \Rightarrow \quad A^{-1} = \begin{bmatrix} -1 & 1 & -4 \\ 0 & -1 & 3 \\ -2 & 3 & -10 \end{bmatrix}.$$

(b)

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 & 1 & -4 \\ 0 & -1 & 3 \\ -2 & 3 & -10 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -2+3-16 \\ -3+12 \\ -4+9-40 \end{bmatrix} = \begin{bmatrix} -15 \\ 9 \\ -35 \end{bmatrix}.$$

#	Score
1	
2	
3	
4	
Σ	

2. (12 points) Consider the matrices

$$A = \begin{bmatrix} -1 & 3 & -2 \\ 2 & 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}.$$

For each of the following expressions, compute it or explain why it is not defined.

(a) A^2 and B^3.

(b) $(BA)^T - 2A^T$.

(c) A^{-1}, and B^{-1}.

(d) Find a 2×3 matrix C such that $BC = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & -1 \end{bmatrix}$.

(a) A^2 not possible because A is 2×3.

$$B^3 = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ -3 & 3 \end{bmatrix} = 3\begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}.$$

(b)

$$BA = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 3 & -2 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 7 & -1 \\ 5 & 5 & 3 \end{bmatrix}.$$

$$(BA)^T - 2A = \begin{bmatrix} 1 & 5 \\ 7 & 5 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 6 & 8 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & -3 \\ 3 & 1 \end{bmatrix}.$$

(c) A^{-1} is not possible because A is not square.

$$B^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}.$$

(d)

$$C = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 & 2 & 3 \\ 3 & 7 & 0 \end{bmatrix}.$$

3. (10 points) Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 3 & -2 \\ 3 & 1 & 2 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix}.$$

(a) Find all the source vectors $\mathbf{b}$ such that the non-homogeneous system $A\mathbf{x} = \mathbf{b}$ is consistent, and write these $\mathbf{b}$ in parametric form. Justify your answer.

(b) Find the null space of A. Justify your answer.

(a)

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & -2 & b_1 \\ 3 & 1 & 2 & 1 & b_2 \\ 0 & -1 & 1 & -1 & b_3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & 3 & -2 & b_1 \\ 0 & 7 & -7 & 7 & b_2 - 3b_1 \\ 0 & -1 & 1 & -1 & b_3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & 3 & -2 & b_1 \\ 0 & -1 & 1 & -1 & b_3 \\ 0 & 7 & -7 & 7 & b_2 - 3b_1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & -2 & 3 & -2 & b_1 \\ 0 & -1 & 1 & -1 & b_3 \\ 0 & 0 & 0 & 0 & b_2 - 3b_1 + 7b_3 \end{array}\right] \quad \Rightarrow \quad b_2 - 3b_1 + 7b_3 = 0.$$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ 3b_1 - 7b_3 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} b_1 + \begin{bmatrix} 0 \\ -7 \\ 1 \end{bmatrix} b_3.$$

(b)

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -x_3, \quad x_2 = x_3 - x_4, \quad x_3, x_4 \text{ free.}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} x_4.$$

4. (10 point) Let W be the subspace of $\mathbb{R}^4$ of vectors of the form

$$W = \left\{ \begin{bmatrix} 3x+2y \\ 0 \\ 2x+y+z \\ -4z-2x \end{bmatrix}, \quad x, y, z \in \mathbb{R} \right\}.$$

(a) Find a set of vectors in $\mathbb{R}^4$ whose span is W. Justify your answer.

(b) Find a basis for W, that is, a l.i. set of vectors in W that spans W. Justify your answer.

(a)

$$\mathbf{w} = \begin{bmatrix} 3x+2y \\ 0 \\ 2x+y+z \\ -4z-2x \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ -2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix} z.$$

$$W = \text{Span} \left\{ \begin{bmatrix} 3 \\ 0 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix} \right\}$$

(b)

$$\begin{bmatrix} 3 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 1 \\ -2 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 2/3 & 0 \\ 2 & 1 & 1 \\ -2 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2/3 & 0 \\ 0 & -1/3 & 1 \\ 0 & 4/3 & -4 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2/3 & 0 \\ 0 & -1/3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the l.i. vectors are

$$\left\{ \begin{bmatrix} 3 \\ 0 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$