Slide 1

Triple integrals

- On rectangular boxes. (Sec. 15.7)
- On simple domains, type I, II, and III.
- On arbitrary domains.

Slide 2

Recall the Riemann sums and their limits

Single variable functions:

$$\lim_{n\to\infty}\sum_{i=0}^{n} f(x_i^*)\Delta x = \int_{x_0}^{x_1} f(x)dx.$$

Two variable functions:

$$\lim_{n\to\infty}\sum_{i=0}^{n}\sum_{j=0}^{n} f(x_i^*, y_j^*)\Delta x\Delta y = \int_{x_0}^{x_1}\int_{y_0}^{y_1} f(x,y)dxdy.$$

Three variable functions:

$$\lim_{n\to\infty}\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} f(x_i^*, y_j^*, z_k^*)\Delta x\Delta y\Delta z = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1} f(x,yz)\, dxdydz.$$

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Integrals in a rectangular box domain

Theorem 1 *Let $f(x, y, z)$ be a continuous function on a rectangular boxed domain $R = [x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]$. Then,*

$$\int\int\int_R f\, dV = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} f(x, y, z)\, dzdydx.$$

Furthermore, the integral does not change when performed in different order.

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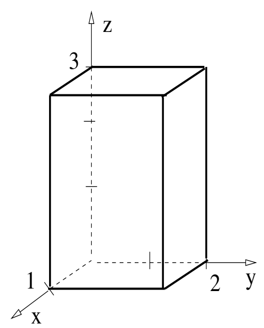
Compute the integral of $f(x, y, z) = xyz^2$ on the domain $R = [0, 1] \times [0, 2] \times [0, 3]$

$$R = \{(x, y, z) \in \mathbb{R}^3 : \ 0 \leqslant x \leqslant 1, \ 0 \leqslant y \leqslant 2, \ 0 \leqslant z \leqslant 3\}.$$



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Notice the order of the integrations

$$\begin{aligned}\int\int\int_R f\,dV &= \int_0^1\int_0^2\int_0^3 xyz^2\,dzdydx,\\ &= \int_0^1\int_0^2 xy\frac{1}{3}\left(z^3\Big|_0^3\right)\,dydx,\\ &= \frac{27}{3}\int_0^1\int_0^2 xy\,dydx,\\ &= 9\int_0^1 x\frac{1}{2}\left(y^2\Big|_0^2\right)\,dx,\\ &= 18\int_0^1 x\,dx,\\ &= 9.\end{aligned}$$

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Triple integrals on simple regions

Type I, means arbitrary shape only on the x variable.

Type II means arbitrary shape only on the y variable.

Type III means arbitrary shape only on the z variable.

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For example, consider an integral type III

Theorem 2 *Let $g_0(x,y) \leqslant g_1(x,y)$ be two continuous functions defined on a domain $[x_0,x_1] \times [y_0,y_1]$. Let $f(x,y,z)$ be a continuous function in*

$$D = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{array}{c} x_0 \leqslant x \leqslant x_1, \\ y_0 \leqslant y \leqslant y_1, \\ g_0(x,y) \leqslant z \leqslant g_1(x,y) \end{array} \right\}.$$

Then, the integral of $f(x,y,z)$ in D is given by

$$\int\int\int_D f(x,y)\,dxdy = \int_{x_0}^{x_1}\int_{y_0}^{y_1}\left[\int_{g_0(x,y)}^{g_1(x,y)} f(x,y,z)\,dz\right]dydx.$$

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Triple integrals on arbitrary domains

Theorem 3 *Let $g_0(x,y) \leqslant g_1(x,y)$ be two continuous functions defined on a domain $[x_0,x_1] \times [y_0,y_1]$. Let $h_0(x) \leqslant h_1(x)$ be two continuous functions defined on a domain $[x_0,x_1]$. Let $f(x,y,z)$ be a continuous function in*

$$D = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{array}{c} x_0 \leqslant x \leqslant x_1, \\ h_0(x) \leqslant y \leqslant h_1(x), \\ g_0(x,y) \leqslant z \leqslant g_1(x,y) \end{array} \right\}.$$

Then, the integral of $f(x,y,z)$ in D is given by

$$\int\int\int_D f\,dV = \int_{x_0}^{x_1}\left[\int_{h_0(x)}^{h_1(x)}\left(\int_{g_0(x,y)}^{g_1(x,y)} f(x,y,z)\,dz\right)dy\right]dx.$$

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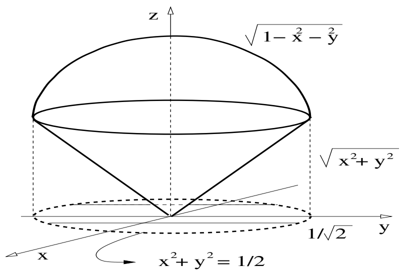
Find the volume between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$



$$V = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1/2-x^2}}^{\sqrt{1/2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dz\, dy\, dx.$$

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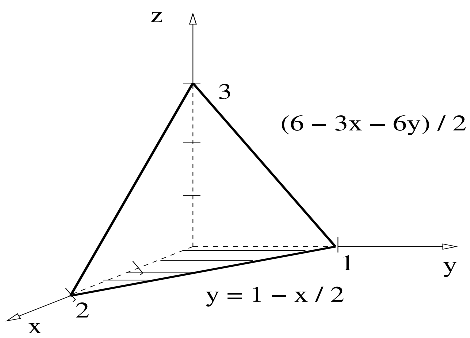
Compute the volume of the region given by $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ and $3x + 6y + 2z \leqslant 6$



$$V = \int_0^2 \left[\int_0^{1-x/2} \left[\int_0^{(6-6y-3x)/2} dz \right] dy \right] dx.$$

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Compute the volume of the region given by $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ and $3x + 6y + 2z \leqslant 6$

$0 \leqslant z \leqslant (6 - 6y - 3x)/2$.
Then, for $z = 0$ one has that $0 \leqslant y \leqslant 1 - x/2$.
Then, for $z = 0$, $y = 0$, one has that $0 \leqslant x \leqslant 2$.

$$V = \int\int\int_D dV = \int_0^2 \left[\int_0^{1-x/2} \left(\int_0^{(6-6y-3x)/2} dz \right) dy \right] dx,$$
$$= 3 \int_0^2 \left[\int_0^{1-x/2} \left(1 - y - \frac{x}{2}\right) dy \right] dx,$$

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Continuation of the example

$$V = 3 \int_0^2 \left[\int_0^{1-x/2} \left(1 - \frac{x}{2} - y\right) dy \right] dx,$$
$$= 3 \int_0^2 \left[\left(1 - \frac{x}{2}\right)\left(1 - \frac{x}{2}\right) - \frac{1}{2}\left(1 - \frac{x}{2}\right)^2 \right] dx,$$
$$= \frac{3}{2} \int_0^2 \left(1 - \frac{x}{2}\right)^2 dx.$$

Then, substitute $u = 1 - x/2$, then $du = -dx/2$, so

$$V = 3 \int_0^1 u^2 du, \quad \Rightarrow \quad \boxed{V = 1}.$$

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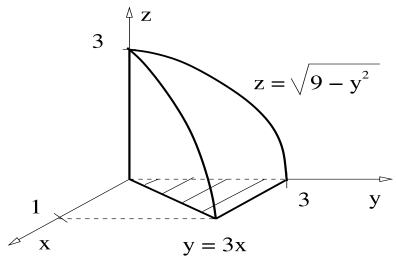
Compute the triple integral of $f(x, y, z) = z$ in the region $0 \leqslant x$, $3x \leqslant y$, $0 \leqslant z$ and $y^2 + z^2 \leqslant 9$



$$V = \int_0^1 \left[\int_{3x}^3 \left(\int_0^{\sqrt{9-y^2}} f(x, y, z)\, dz \right) dy \right] dx.$$

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Compute the triple integral of $f(x, y, z) = z$ in the region $0 \leqslant x$, $3x \leqslant y$, $0 \leqslant z$ and $y^2 + z^2 \leqslant 9$

$$\begin{aligned}
\int\int\int_D f\, dv &= \int_0^1 \left[\int_{3x}^3 \left(\int_0^{\sqrt{9-y^2}} z\, dz \right) dy \right] dx, \\
&= \int_0^1 \left[\int_{3x}^3 \frac{1}{2} \left(z^2 \Big|_0^{\sqrt{9-y^2}} \right) dy \right] dx, \\
&= \frac{1}{2} \int_0^1 \left[\int_{3x}^3 (9 - y^2) dy \right] dx, \\
&= \frac{1}{2} \int_0^1 \left[27(1 - x) - \frac{1}{3} \left(y^3 \Big|_{3x}^3 \right) \right] dx, \\
&= \frac{1}{2} \int_0^1 \left[27(1 - x) - 9(1 - x)^3 \right] dx, \\
&= \frac{9}{2} \int_0^1 \left[3(1 - x) - (1 - x)^3 \right] dx.
\end{aligned}$$

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Continuation of the example

Substitute $u = 1 - x$, then $du = -dx$, so,

$$\int\int\int_D f\,dv = \frac{9}{2}\int_0^1 (3u - u^3)du,$$
$$= \frac{9}{2}\left[\frac{3}{2}\left(u^2\big|_0^1\right) - \frac{1}{4}\left(u^4\big|_0^1\right)\right],$$
$$= \frac{9}{2}\left(\frac{3}{2} - \frac{1}{4}\right),$$
$$= \frac{45}{8}, \quad \Rightarrow \quad \boxed{\int\int\int_D f\,dv = \frac{45}{8}}.$$

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Cylindrical and spherical coordinates

- Review of triple integrals in Cartesian coordinates.
- Cylindrical and spherical coordinates in $\mathbb{R}^3$. (Sec. 12.7)
- Triple integral in cylindrical and spherical coordinates. (Sec. 15.8)

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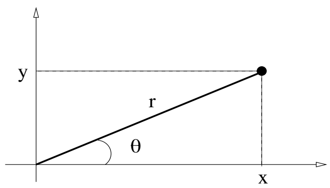
Recall polar coordinates in $\mathbb{R}^2$

Definition 1 *Let (x, y) be Cartesian coordinates in $\mathbb{R}^2$. Then, polar coordinates (r, θ) are defined in $\mathbb{R}^2 - \{(0,0)\}$, and given by*

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

The inverse expression is

$$x = r\cos(\theta),$$
$$y = r\sin(\theta).$$



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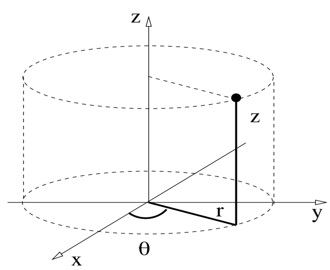
Cylindrical coordinates are polar coord. plus z

Definition 2 Let (x, y, z) be Cartesian coordinates in $\mathbb{R}^3$. Then, cylindrical coordinates (r, θ, z) are defined in $\mathbb{R}^3 - \{(0,0,z)\}$, and given by

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right), \quad z = z.$$

$$x = r\cos(\theta),$$
$$y = r\sin(\theta),$$
$$z = z.$$



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From Riemann sums in cylindrical coordinates one gets the following formula for triple integrals

$$\int\int\int_R f\,dV = \int\int\int_R f(x,y,z)\,(r\,dr)\,d\theta\,dz.$$

where in $f(x,y,z)$ one has to replace

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z.$$

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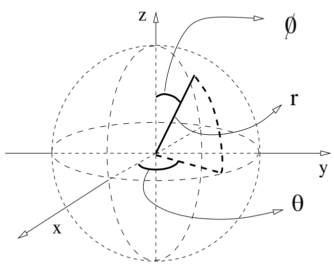
Spherical coordinates in $\mathbb{R}^3$

Definition 3 Let (x,y,z) be Cartesian coordinates in $\mathbb{R}^3$. Then, spherical coordinates (r,θ,ϕ) are defined in $\mathbb{R}^3 - \{(0,0,0)\}$, and given by

$$r = \sqrt{x^2+y^2+z^2}, \quad \theta = \arctan\left(\frac{y}{x}\right), \quad \phi = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right).$$

$$x = r\sin(\phi)\cos(\theta),$$
$$y = r\sin(\phi)\sin(\theta),$$
$$z = r\cos(\phi).$$



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From Riemann sums in spherical coordinates one gets the following formula for triple integrals

$$\int\int\int_R f\,dV = \int\int\int_R f(x,y,z)\,(r^2dr)\,(\sin(\phi)\,d\phi)\,d\theta.$$

where in $f(x,y,z)$ one has to replace

$$\begin{aligned} x &= r\sin(\phi)\cos(\theta), \\ y &= r\sin(\phi)\sin(\theta), \\ z &= r\cos(\phi). \end{aligned}$$

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Find the volume of a cylinder of radius R and height h

$$\begin{aligned} V &= \int_0^{2\pi}\int_0^R\int_0^h dz\,(r\,dr)\,d\theta, \\ &= h\int_0^{2\pi}\int_0^R r\,dr\,d\theta, \\ &= h\frac{R^2}{2}\int_0^{2\pi} d\theta, \\ &= h\frac{R^2}{2}2\pi, \\ &\boxed{V = \pi R^2 h}. \end{aligned}$$

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Find the volume of a cone of base radius r_0 and height h_0

$$
\begin{aligned}
V &= \int_0^{2\pi}\int_0^{R}\int_0^{h(1-r/R)} dz\,(r\,dr)\,d\theta,\\
&= h\int_0^{2\pi}\int_0^{R}\Big(1-\frac{r}{R}\Big)r\,dr\,d\theta,\\
&= h\int_0^{2\pi}\int_0^{R}\Big(r-\frac{r^2}{R}\Big)\,dr\,d\theta,\\
&= h\Big(\frac{R^2}{2}-\frac{R^3}{3R}\Big)\int_0^{2\pi} d\theta,\\
&= 2\pi h\frac{R^2}{2}\Big(1-\frac{2}{3}\Big), \quad\Rightarrow\quad \boxed{V=\frac{1}{3}\pi R^2 h}.
\end{aligned}
$$

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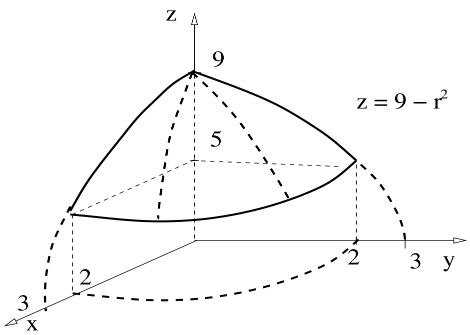
Find the solid whose volume is given by

$V = \int_0^{\pi/2}\int_0^{2}\int_0^{9-r^2} r\,dz\,dr\,d\theta$ (Sec. 15.8, Probl. 2)



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Find the volume of a sphere of radius R

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} (r^2\,dr)\,(\sin(\phi)\,d\phi)\,d\theta,$$

$$= \left[\int_0^{2\pi} d\theta\right]\left[\int_0^{\pi}\sin(\phi)\,d\phi\right]\left[\int_0^{R} r^2\,dr\right],$$

$$= 2\pi\left(-\cos(\phi)\Big|_0^{\pi}\right)\frac{R^3}{3},$$

$$= 2\pi\left(-\cos(\pi)+\cos(0)\right)\frac{R^3}{3} \quad\Rightarrow\quad \boxed{V = \frac{4}{3}\pi R^3}.$$

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Exercises

- Find the volume below the sphere $x^2 + y^2 + z^2 = 1$ and above the cone $z = \sqrt{x^2 + y^2}$. (Answer: $V = \pi(2 - \sqrt{2})/3$.)
- Find the volume below the sphere $x^2 + y^2 + z^2 = z$ and above the cone $z = \sqrt{x^2 + y^2}$. (Answer: $V = \pi/8$.)

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Exercises

- (Probl. 20, Sec. 15.8) Compute the integral
$$I = \int\int\int_R e^{\sqrt{x^2+y^2+z^2}} dV,$$
where the region R is the portion within the sphere $x^2 + y^2 + z^2 = 9$ in the fist octant.
(Answer: $I = \pi(5e^3 - 1)/2$.)
- (Probl. 35, Sec. 15.8) Change to spherical coordinates and compute the following integral,
$$I = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} x\sqrt{x^2 + y^2 + z^2} dz\, dy\, dx.$$
(Answer: $I = 3^5\pi/5$.)