**Slide 1**

**Double integrals (Sec. 15.1 - 15.2)**

- Review of the integral of single variable functions.
- Definition of a double integral on rectangles.
- Average of a function.
- Examples of double integrals in rectangles (sec. 15.2)

**Slide 2**

**Integral of a single variable function**

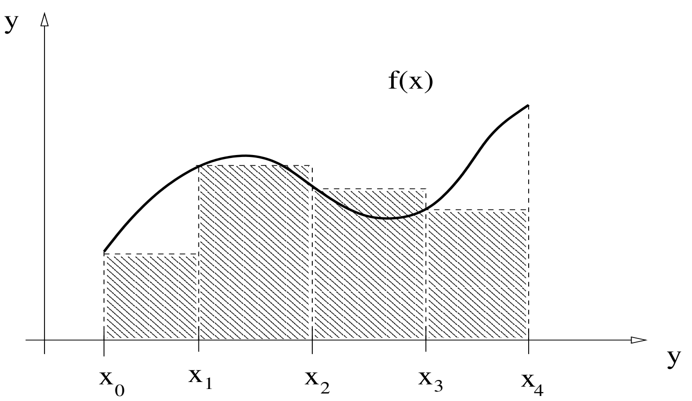
**Definition 1** *Let $f(x)$ be a function defined on a interval $x \in [a, b]$. The integral of $f(x)$ in $[a, b]$ is the number given by*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i^*)\, \Delta x,$$

*if the limit exists. Given a natural number $n$ we have introduced a partition on $[a, b]$ given by $\Delta x = (b - a)/n$. We denoted $x_i^* = (x_i + x_{i-1})/2$, where $x_i = a + i\Delta x$, $i = 0, 1, \cdots, n$. This choice of the sample point $x_i^*$ is called midpoint rule.*

**Slide 3**

## Riemann sum of a single variable function




**Slide 4**

## Double integrals on rectangles

**Definition 2** *Let $f(x,y)$ be a function defined on a rectangle $R = [x_0, x_1] \times [y_0, y_1]$. The integral of $f(x,y)$ in $R$ is the number given by*

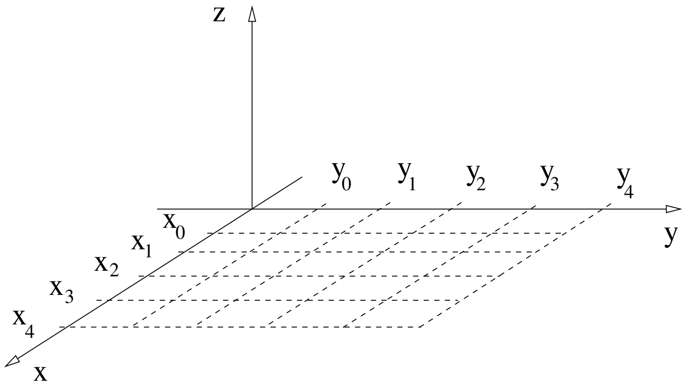
$$\int\int_R f(x)\,dxdy = \lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*)\,\Delta x \Delta y,$$

*if the limit exists.*

Given a natural number $n$, the partition on $R$ are rectangles of side $\Delta x = (x_1 - x_0)/n$, $\Delta y = (y_1 - y_0)/n$. Let $x_i^* = (x_i + x_{i-1})/2$, $y_j^* = (y_j + y_{j-1})/2$, where $x_i = x_0 + i\Delta x$, and $y_j = y_0 + j\Delta y$, for $i, j = 0 \cdots, n$. These sample points $x_i^*$, $y_j^*$ are called midpoint rule.

**Slide 5**

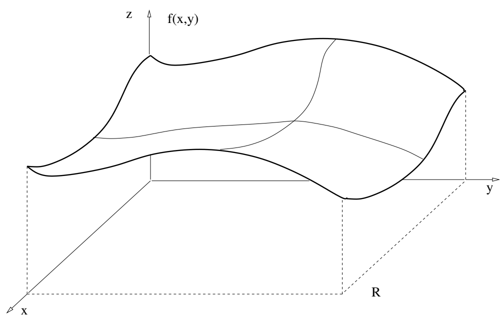
## Partition of the domain of a two variable function




**Slide 6**

## Double integrals of $f(x, y)$ are volumes in $\mathbb{R}^3$

If $f(x, y) \geq 0$, then $\int\int_R f(x, y)\, dxdy = V$ the volume above $R$ and below the surface given by the graph of $f(x, y)$.



**Slide 7**

## The order of integration can be switched in double integrals of continuous functions

**Theorem 1 (Fubini)** *If $f(x,y)$ is a continuous function in $R = [x_0, x_1] \times [y_0, y_1]$, then*

$$\int\int_R f(x,y)\,dxdy = \int_{y_0}^{y_1} \left[\int_{x_0}^{x_1} f(x,y)\,dx\right] dy,$$
$$= \int_{x_0}^{x_1} \left[\int_{y_0}^{y_1} f(x,y)\,dy\right] dx.$$

Notation: One also denotes the double integral as

$$\int\int_R f(x,y)\,dxdy = \int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)\,dxdy.$$

**Slide 8**

## Here is an example of a double integral

$$\int_1^3\int_0^2 (xy^2 + 2x^2y^3)dxdy =$$
$$= \int_1^3 \left[\int_0^2 (xy^2 + 2x^2y^3)dx\right] dy,$$
$$= \int_1^3 \left[\frac{1}{2}y^2\left(x^2\big|_0^2\right) + \frac{2}{3}y^3\left(x^3\big|_0^2\right)\right] dy,$$
$$= \int_1^3 \left[2y^2 + \frac{16}{3}y^3\right] dy,$$
$$= \frac{2}{3}y^3\Big|_1^3 + \frac{16}{12}y^4\Big|_1^3,$$
$$= \frac{2}{3}26 + \frac{4}{3}80.$$

**Slide 9**

## Second example

$$\begin{aligned}
\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy dx &= \int_1^4 \left[\int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy\right] dx, \\
&= \int_1^4 \left[x \left(\ln(y)\big|_1^2\right) + \frac{1}{2x}\left(y^2\big|_1^2\right)\right] dx, \\
&= \int_1^4 \left[\ln(2)x + \frac{3}{2x}\right] dx, \\
&= \ln(2)\frac{1}{2}\, x^2\big|_1^4 + \frac{3}{2}\ln(x)\big|_1^4, \\
&= \frac{15}{2}\ln(2) + \frac{3}{2}\ln(4), \\
&= \left(\frac{15}{2} + 3\right)\ln(2).
\end{aligned}$$

**Slide 10**

## Fubini theorem in the case of $f(x,y) = g(x)h(y)$:

$$\int_{x_0}^{x_1} \int_{y_0}^{y_1} g(x)h(y) dy dx = \left(\int_{x_0}^{x_1} g(x) dx\right)\left(\int_{y_0}^{y_1} h(y) dy\right).$$

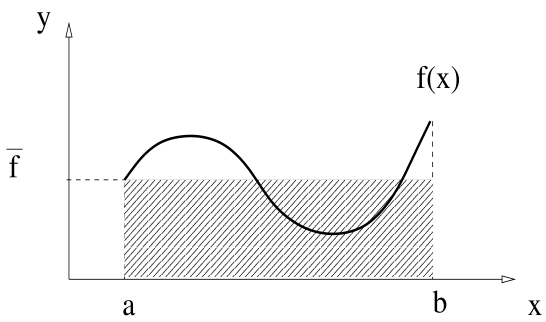
Example:

$$\begin{aligned}
\int_0^2 \int_0^1 \frac{1+x^2}{1+y^2} dy dx &= \left[\int_0^2 (1+x^2) dx\right]\left[\int_0^1 \frac{1}{1+y^2} dy\right], \\
&= \left(x\big|_0^2 + \frac{1}{3}x\big|_0^2\right)\left(\arctan(y)\big|_0^1\right), \\
&= \frac{\pi}{4}\left(2 + \frac{8}{3}\right).
\end{aligned}$$

**Slide 11**

**Recall the average of $f(x)$ in $[a, b]$**

The number $\overline{f}$ given by $\overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$ is the average of $f(x)$ in $[a, b]$.




**Slide 12**

**The average of $f(x, y)$ in $R$**

**Definition 3 (Average)** *The number $\overline{f}$ given by*

$$\overline{f} = \frac{1}{A(R)}\int_R f(x, y)\,dxdy,$$

*is the average of a function $f(x, y)$ in the domain $R = [x_0, x_1] \times [y_0, y_1]$, where*

$$A(R) = (x_1 - x_0)(y_1 - y_0)$$

*the area of the rectangle domain $R$.*

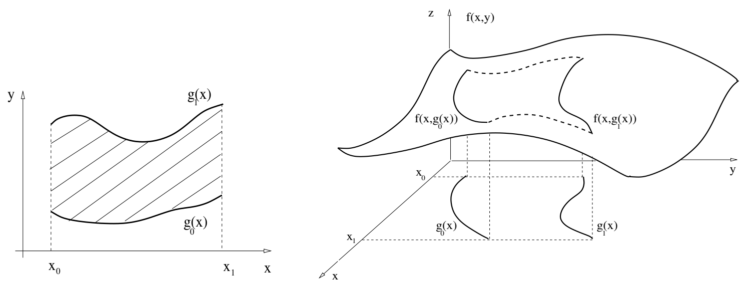
**Slide 13**

## Double integrals on regions

- Regions in Cartesian coordinates (Sec. 15.3)
  - Type I: Regions functions $y(x)$.
  - Type II: Regions functions $x(y)$.
- Regions in Cartesian coordinates (Sec. 15.4)
  - Type I: Regions functions $r(\theta)$.
  - Type II: Regions functions $\theta(r)$.

**Slide 14**

## Regions in Cartesian coordinates $y(x)$: Type I



**Slide 15**

**Regions in Cartesian coordinates $y(x)$: Type I**

**Theorem 2** *Let $g_0(x)$, $g_1(x)$ be two continuous functions defined on an interval $[x_0, x_1]$, and such that $g_0(x) \leq g_1(x)$. Let $f(x,y)$ be a continuous function in*

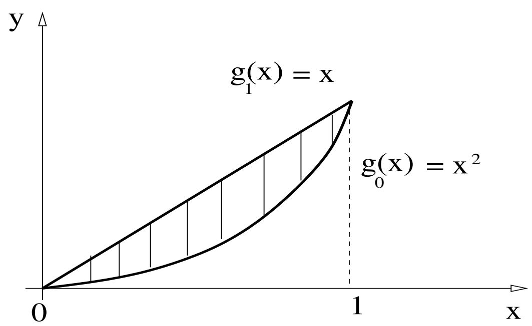
$$D = \{(x,y) \in I\!R^2 : x_0 \leq x \leq x_1, \quad g_0(x) \leq y \leq g_1(x)\}.$$

*Then, the integral of $f(x,y)$ in $D$ is given by*

$$\int\int_D f(x,y)\,dxdy = \int_{x_0}^{x_1} \left[\int_{g_0(x)}^{g_1(x)} f(x,y)dy\right] dx.$$

**Slide 16**

**Cartesian Type I: Find the $\int\int_D f(x,y)\,dxdy$ for $f(x,y) = x^2 + y^2$, on** $D = \{(x,y) \in I\!R^2 : 0 \leq x \leq 1, \quad x^2 \leq y \leq x\}$
.

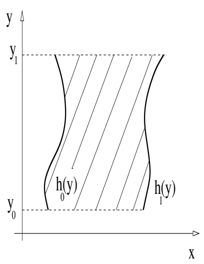


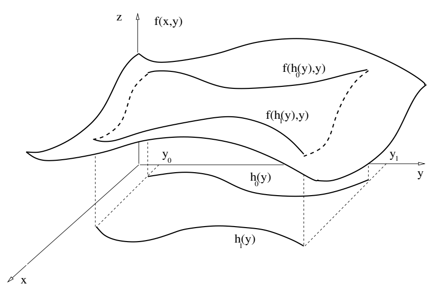
**Slide 17**

$$\begin{aligned}
\int\int_D f(x,y)\,dxdy &= \int_0^1 \left[\int_{x^2}^{x} (x^2+y^2)dy\right] dx,\\
&= \int_0^1 \left[x^2\left(y|_{x^2}^{x}\right) + \frac{1}{3}\left(y^3|_{x^2}^{x}\right)\right] dx,\\
&= \int_0^1 \left[x^2(x-x^2) + \frac{1}{3}(x^3-x^6)\right] dx,\\
&= \int_0^1 \left[x^3 - x^4 + \frac{1}{3}x^3 - \frac{1}{3}x^6\right] dx,\\
&= \left[\frac{1}{4}x^4 - \frac{1}{5}x^5 + \frac{1}{12}x^4 - \frac{1}{21}x^7\right]\Big|_0^1,\\
&= \frac{1}{3} - \frac{1}{5} - \frac{1}{21} = \frac{9}{3\times 5\times 7}.
\end{aligned}$$

**Slide 18**

## Regions in Cartesian coordinates $x(y)$: Type II



**Slide 19**

**Regions in Cartesian coordinates $x(y)$: Type II**

**Theorem 3** *Let $h_0(y)$, $h_1(y)$ be two continuous functions defined on an interval $[y_0, y_1]$, and such that $h_0(y) \leq h_1(y)$. Let $f(x, y)$ be a continuous function in*

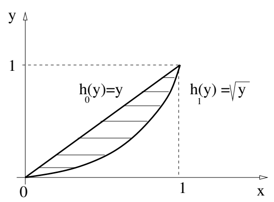
$$D = \{(x, y) \in I\!R^2 : h_0(y) \leq x \leq h_1(y), \quad y_0 \leq y \leq y_1\}.$$

*Then, the integral of $f(x, y)$ in $D$ is given by*

$$\int\int_D f(x, y)\, dxdy = \int_{y_0}^{y_1} \left[ \int_{h_0(y)}^{h_1(y)} f(x, y) dx \right] dy.$$

**Slide 20**

**Cartesian Type II: Find the $\int\int_D f(x, y)\, dxdy$ for $f(x, y) = x^2 + y^2$, on** $D = \{(x, y) \in I\!R^2 : 0 \leq x \leq 1, \quad x^2 \leq y \leq x\}$ .




Notice that $h_0(y) = y$, and $h_1(y) = \sqrt{y}$. Then,

$$D = \{(x, y) \in I\!R^2 : h_0(y) = y \leq x \leq h_1(y) = \sqrt{y}, \quad y_0 \leq y \leq y_1\}.$$

**Slide 21**

$$\int\int_D f(x,y)\,dxdy = \int_0^1 \left[\int_y^{\sqrt{y}} (x^2+y^2)dx\right] dy,$$
$$= \int_0^1 \left[\frac{1}{3}\left(x^3|_y^{\sqrt{y}}\right) + y^2\left(x|_y^{\sqrt{y}}\right)\right] dy,$$
$$= \int_0^1 \left[\frac{1}{3}(y^{3/2}-y^3) + y^2(y^{1/2}-y)\right] dy,$$
$$= \int_0^1 \left[\frac{1}{3}y^{3/2} - \frac{1}{3}y^3 + y^{5/2} - y^3\right] dy,$$
$$= \left[\frac{1}{3}\frac{2}{5}y^{5/2} - \frac{1}{3}\frac{1}{4}y^4 + \frac{2}{7}y^{7/2} - \frac{1}{4}y^4\right]\Big|_0^1,$$
$$= \frac{2}{15} - \frac{1}{12} + \frac{2}{7} - \frac{1}{4} = \frac{9}{3\times 5\times 7}.$$

**Slide 22**

**Find the** $\int\int_D f(x,y)\,dxdy$ **for** $f(x,y)=1$**, and** $D = \left\{(x,y)\in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{4} \le 1\right\}$

As type I, then,

$$g_1(x) = 3\sqrt{1-y^2/4}, \quad g_0(x) = -3\sqrt{1-y^2/4}.$$

As type II, then,

$$h_1(x) = 2\sqrt{1-x^2/9}, \quad h_0(y) = -2\sqrt{1-x^2/9}.$$

**Slide 23**

**Review of polar coordinates**

**Definition 4** *Let* $(x, y)$ *be Cartesian coordinates in* $\mathbb{R}^2$. *Then, polar coordinates* $(r, \theta)$ *are defined in* $\mathbb{R}^2 - \{(0,0)\}$, *and given by*

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

*The inverse expression is*

$$x = r\cos(\theta), \quad y = r\sin(\theta).$$

**Slide 24**

**Double integrals in polar coordinates on disk sections**

**Theorem 4** *If* $f(r, \theta)$ *in continuous in*

$$D = \{(r, \theta) : 0 < r_0 \le r \le r_1, \quad \theta_0 \le \theta \le \theta_1 < 2\pi\},$$

*then* $\displaystyle\int\int_D f(r,\theta)\, dA = \int_{\theta_0}^{\theta_1}\int_{r_0}^{r_1} f(r,\theta)\, r\, dr d\theta.$

**Disk sections in polar coordinates ↔ rectangular sections in Cartesian coordinates**

**Slide 25**

## Compute the integral of $f(x, y) = x^2 + 2y^2$ on

$D = \{(x, y) \in I\!R^2 : 0 \leq y, \quad 0 \leq x, \quad 1 \leq x^2 + y^2 \leq 2\}$

Translate to polar coordinates. $x = r\cos(\theta)$, $y = r\sin(\theta)$. Then

$$f(r, \theta) = r^2 + r^2 \sin^2(\theta).$$

The region $D$ is $D = \{(r, \theta) \in I\!R^2 : 0 \leq \theta \leq \frac{\pi}{2}, \quad 1 \leq r \leq \sqrt{2}\}$.

$$\begin{aligned}\int\int_D f(r, \theta) dA &= \int_0^{\pi/2} \int_1^{\sqrt{2}} r^2 (1 + \sin^2(\theta)) r \, dr d\theta, \\ &= \left[\int_0^{\pi/2} (1 + \sin^2(\theta)) \, d\theta\right] \left[\int_1^{\sqrt{2}} r^3 dr\right],\end{aligned}$$

**Slide 26**

## Example: Continuation

$$\begin{aligned}\int\int_D f(r, \theta) dA &= \left[(\theta|_0^{\pi/2}) + \int_0^{\pi/2} \frac{1}{2}(1 - \cos(2\theta)) \, d\theta\right] \left[\frac{1}{4}(r^4|_1^{\sqrt{2}})\right], \\ &= \left[\frac{\pi}{2} + \frac{1}{2}(\theta|_0^{\pi/2}) - \frac{1}{4}(\sin(2\theta)|_0^{\pi/2})\right] \frac{3}{4}, \\ &= \frac{3}{4}\left[\frac{\pi}{2} + \frac{\pi}{4}\right], \\ &= \frac{9}{16}\pi.\end{aligned}$$

**Slide 27**

**Integrate $f(x,y) = e^{-(x^2+y^2)}$ on**

$D = \{(r,\theta) \in R^2 : \quad 0 \leq \theta \leq \pi, \quad 0 \leq r \leq 2\}$

Notice, $f(r,\theta) = e^{-r^2}$, then,

$$\int\int_D e^{-(x^2+y^2)} dA = \int_0^{\pi} \int_0^2 e^{-r^2} r\, dr d\theta,$$

substitute $u = r^2$, then $du = 2r\, dr$, then

$$\begin{aligned} \int\int_D e^{-(x^2+y^2)} dA &= \frac{1}{2} \int_0^{\pi} \int_0^4 e^{-u} du d\theta \\ &= \frac{1}{2} \int_0^{\pi} (-e^{-u}|_0^4) d\theta, \\ &= \frac{\pi}{2} \left(1 - \frac{1}{e^4}\right). \end{aligned}$$

**Slide 28**

**Summarizing, from Cartesian to polar**

**Theorem 5** *Let $f(x,y)$ be a continuous function on a domain $D$, where $(x,y)$ represent Cartesian coordinates. Let $(r,\theta)$ be polar coordinates. Then the following formula holds,*

$$\int\int_D f(x,y)\, dx dy = \int\int_D f(r\cos(\theta), r\sin(\theta)) r\, dr d\theta.$$

**Slide 29**

**Type I in polar coordinates**

**Theorem 6** *Let* $0 < h_0(\theta) \leq h_1(\theta)$ *be two continuous functions defined on an interval* $[\theta_0, \theta_1]$*. Let* $f(r, \theta)$ *be a continuous function in*

$$D = \{(r, \theta) \in I\!R^2 : 0 < \ h_0(\theta) \leq r \leq h_1(\theta), \\ \theta_0 \leq \theta \leq \theta_1\}.$$

*Then, the integral of* $f(r, \theta)$ *in* $D$ *is given by*

$$\int\int_D f(r, \theta)\, dA = \int_{\theta_0}^{\theta_1} \left[ \int_{h_0(\theta)}^{h_1(\theta)} f(r, \theta) r\, dr \right] d\theta.$$

**Slide 30**

**Type II in polar coordinates**

**Theorem 7** *Let* $g_0(r)$*,* $g_1(r)$ *be two continuous functions defined on an interval* $[r_0, r_1]$*, and such that* $0 < g_0(r) \leq g_1(r) < 2\pi$*. Let* $f(r, \theta)$ *be a continuous function in*

$$D = \{(r, \theta) \in I\!R^2 : 0 < r_0 \leq r \leq r_1, \\ 0 < \ g_0(r) \leq \theta \leq g_1(r) \ < 2\pi\}.$$

*Then, the integral of* $f(r, \theta)$ *in* $D$ *is given by*

$$\int\int_D f(r, \theta)\, dA = \int_{r_0}^{r_1} \left[ \int_{g_0(r)}^{g_1(r)} f(r, \theta) d\theta \right] r dr.$$