Slide 1

Directional derivative and gradient vector

- Definition of directional derivative. (Sec. 14.6)
- Directional derivative and partial derivatives.
- Gradient vector.
- Geometrical meaning of the gradient.

Slide 2

The directional derivative generalizes the partial derivatives to any direction

Definition 1 *The directional derivative of the function $f(x, y)$ at the point (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle u_x, u_y \rangle$ if*

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{t \to 0} \frac{1}{t} \left[f(x_0 + u_x t, y_0 + u_y t) - f(x_0, y_0) \right],$$

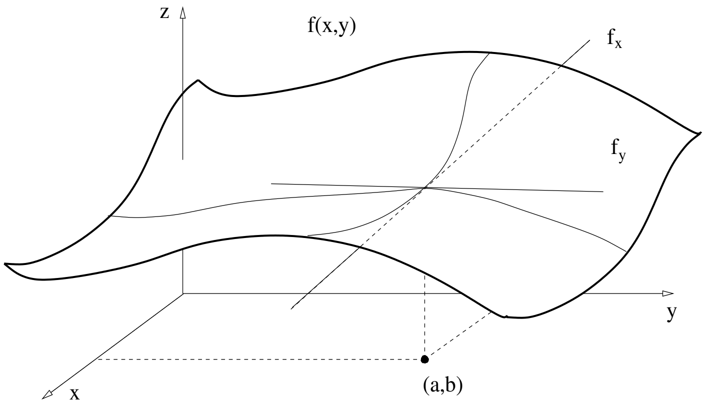
if the limit exists.

Particular cases:

- $\mathbf{u} = \langle 1, 0 \rangle = \mathbf{i}$, then $D_{\mathbf{i}} f(x_0, y_0) = f_x(x_0, y_0)$.
- $\mathbf{u} = \langle 0, 1 \rangle = \mathbf{j}$, then $D_{\mathbf{j}} f(x_0, y_0) = f_y(x_0, y_0)$.

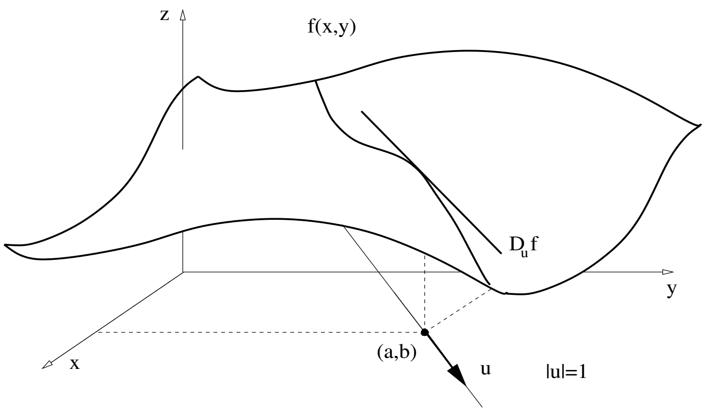
Slide 3

Recall the definition of partial derivatives



Slide 4

The directional derivative generalizes the partial derivatives to any direction



Slide 5

$|\mathbf{u}| = 1$ **implies that** t **is the distance between the points** $(x, y) = (x_0 + u_x t, y_0 + u_y t)$ **and** (x_0, y_0)

$$\begin{aligned} d &= |\langle x - x_0, y - y_0 \rangle|, \\ &= |\langle u_x t, u_y t \rangle|, \\ &= |t|\,|\mathbf{u}|, \\ &= |t|. \end{aligned}$$

The directional derivative of $f(x, y)$ at (x_0, y_0) along $\mathbf{u}$ is the pointwise rate of change of f with respect to the distance along the line parallel to $\mathbf{u}$ passing through (x_0, y_0).

Slide 6

Here is a useful formula to compute directional derivative

Theorem 1 *If* $f(x, y)$ *is differentiable and* $\mathbf{u} = \langle u_x, u_y \rangle$ *is a unit vector, then*

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)\, u_x + f_y(x_0, y_0)\, u_y.$$

The proof is based in the chain rule, case 1

Slide 7

Proof of the theorem

Chain rule case 1, for $x(t) = x_0 + u_x t$, $y(t) = y_0 + u_y t$. Then, $z(t) = f(x(t), y(t))$.

On the one hand,

$$\begin{aligned} \left.\frac{dz}{dt}\right|_{t=0} &= \lim_{t\to 0} \frac{1}{t}[z(t) - z(0)], \\ &= \lim_{t\to 0} \frac{1}{t}[f(x_0 + u_x t, y_0 + u_y t) - f(x_0, y_0)], \\ &= D_{\mathbf{u}} f(x_0, y_0). \end{aligned}$$

Slide 8

Proof of the theorem (Cont.)

On the other hand,

$$\begin{aligned} \frac{dz}{dt}(t) &= f_x(x(t), y(t)) \frac{dx}{dt}(t) + f_y(x(t), y(y)) \frac{dy}{dt}(t), \\ &= f_x(x(t), y(t)) u_x + f_y(x(t), y(t)) u_y, \end{aligned}$$

then,

$$\left.\frac{dz}{dt}\right|_{t=0} = f_x(x_0, y_0) u_x + f_y(x_0, y_0) u_y.$$

Therefore,

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0) u_x + f_y(x_0, y_0) u_y.$$

□

Slide 9

Example about how to compute a directional derivative

Let $f(x, y) = \sin(x + 2y)$. Compute the directional derivative of $f(x, y)$ at $(4, -2)$ in the direction $\theta = \pi/6$.

$$\mathbf{u} = \langle \cos(\theta), \sin(\theta) \rangle, \quad \mathbf{u} = \langle \sqrt{3}/2, 1/2 \rangle.$$

Also

$$f_x = \cos(x + 2y), \quad f_y = 2\cos(x + 2y),$$

then

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= \cos(x + 2y) u_x + 2\cos(x + 2y) u_y, \\ D_{\mathbf{u}} f(4, -2) &= \frac{\sqrt{3}}{2} + 1. \end{aligned}$$

Slide 10

Directional derivatives can be defined on functions of 2, 3 or more variables

Definition 2 (functions of 3 variables)
The directional derivative of the function $f(x, y, z)$ at the point (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle u_x, u_y, u_z \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{t \to 0} \frac{1}{t} \left[f(x_0 + u_x t, y_0 + u_y t, z_0 + u_z t) - f(x_0, y_0, z_0) \right],$$

if the limit exists.

Slide 11

The same useful theorem we had in 2 variable functions

Theorem 2 *If* $f(x, y, z)$ *is differentiable and* $\mathbf{u} = \langle u_x, u_y, u_z \rangle$ *is a unit vector, then*

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)\, u_x + f_y(x_0, y_0, z_0)\, u_y + f_z(x_0, y_0, z_0) u_z.$$

Slide 12

The directional derivative can be written in terms of a dot product

In the case of 2 variable functions:

$$D_{\mathbf{u}} f = f_x u_x + f_y u_y = (\nabla f) \cdot \mathbf{u},$$

with $\nabla f = \langle f_x, f_y \rangle$.

In the case of 3 variable functions:

$$D_{\mathbf{u}} f = f_x u_x + f_y u_y + f_z u_z = (\nabla f) \cdot \mathbf{u},$$

with $\nabla f = \langle f_x, f_y, f_z \rangle$.

Slide 13

We introduce the gradient vector for functions of 2 or 3 variables

Definition 3 *Let $f(x, y, z)$ be a differentiable function. Then,*

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle,$$

is called the gradient of $f(x, y, z)$.

In 2 variables: $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$.

Alternative notation: $\nabla f = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k}$.

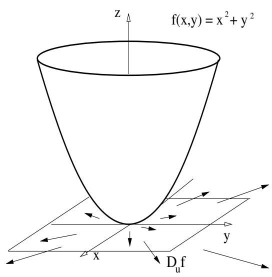
Slide 14

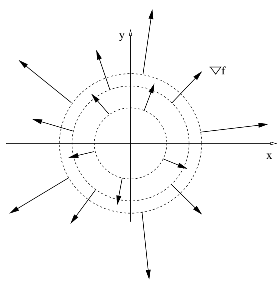
The useful theorem now has the following form

Theorem 3 *Let $f(x, y, z)$ be differentiable function. Then,*

$$D_{\mathbf{u}} f(\mathbf{x}) = (\nabla f(\mathbf{x})) \cdot \mathbf{u}.$$

with $|\mathbf{u}| = 1$.





Slide 15

Gradient vector The gradient vector has two main properties:

- It points in the direction of the maximum increase of f, and $|\nabla f|$ is the value of the maximum increase rate.
- ∇f is normal to the level surfaces.

Slide 16

Here is the first property of the gradient vector

Theorem 4 *Let f be a differentiable function of 2 or 3 variables. Fix $P_0 \in D(f)$, and let $\mathbf{u}$ be an arbitrary unit vector.*

Then, the maximum value of $D_{\mathbf{u}} f(P_0)$ among all possible directions is $|\nabla f(P_0)|$, and it is achieved for $\mathbf{u}$ parallel to $\nabla f(P_0)$.

Slide 17

The proof of the first property

$$\begin{aligned} D_{\mathbf{u}}f(P_0) &= (\nabla f(P_0))\cdot \mathbf{u}, \\ &= |\nabla f(P_0)|\,|\mathbf{u}|\cos(\theta), \\ &= |\nabla f(P_0)|\cos(\theta). \end{aligned}$$

But $-1 \leq \cos(\theta) \leq 1$ implies

$$-|\nabla f(P_0)| \leq D_{\mathbf{u}}f(P_0) \leq |\nabla f(P_0)|.$$

And $D_{\mathbf{u}}f(P_0) = |\nabla f(P_0)|$, $\Leftrightarrow \theta = 0 \Leftrightarrow \mathbf{u}$ is parallel $\nabla f(P_0)$. □

Slide 18

Here is the second property of the gradient vector, in the case of 3 variable functions

Theorem 5 *Let $f(x,y,z)$ be a differentiable at P_0. Then, $\nabla f(P_0)$ is orthogonal to the plane tangent to a level surface containing P_0.*

Slide 19

Proof of the second property

Let $\mathbf{r}(t)$ be any differentiable curve in the level surface $f(x,y,z) = k$. Assume that $\mathbf{r}(t=0) = \overrightarrow{OP}_0$. Then,

$$\begin{aligned} 0 &= \frac{df}{dt}, \\ &= f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}, \\ &= \nabla f(\mathbf{r}(t)) \cdot \frac{dx\mathbf{r}}{dt}(t). \end{aligned}$$

But $(d\mathbf{r})/(dt)$ is tangent to the level surface for any choice of $\mathbf{r}(t)$. Therefore

$$\nabla f(\mathbf{r}(t=0)) \cdot \frac{\mathbf{r}}{dt}(t=0) = 0$$

implies that $\nabla f(P_0)$ is orthogonal to the level surface. □

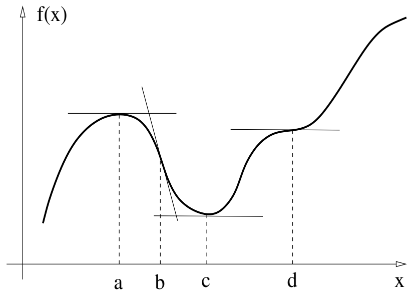
Slide 20

Local and absolute maxima, minima, and inflection points

- Definitions of local extrema. (Sec. 14.7)
- Characterization of local extrema.
- Absolute extrema on closed and bounded sets.
- Typical exercises.

Slide 21

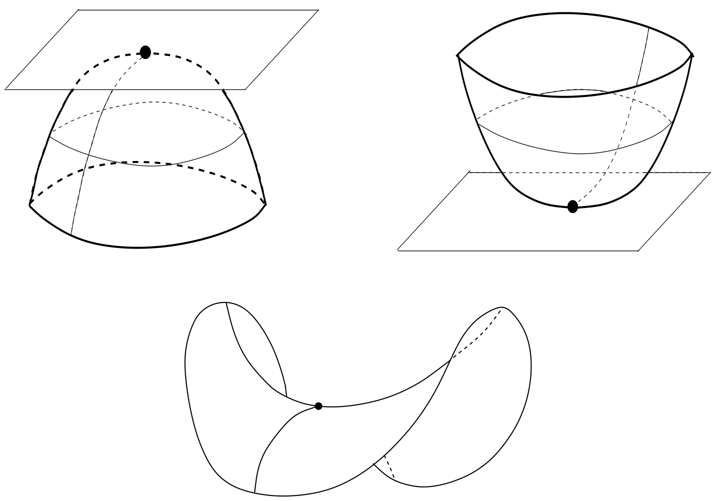
Recall the main results on local extrema for $f(x)$



at	f	f'	f''
a	max.	0	< 0
b	infl.	$\neq 0$	$\pm 0 \mp$
c	min.	0	> 0
d	infl.	$= 0$	$\pm 0 \mp$

Slide 22

The main cases of local extrema for $f(x, y)$



Slide 23

The intuitive notions of local extrema can be written precisely as follows

Definition 4 (Local maximum) *A function* $f(x, y)$ *has a local maximum at* $(a, b) \in D(f) \Leftrightarrow f(x, y) \leq f(a, b)$ *for all* (x, y) *near* (a, b).

Definition 5 (Local minimum) *A function* $f(x, y)$ *has a local minimum at* $(a, b) \in D(f) \Leftrightarrow f(x, y) \geq f(a, b)$ *for all* (x, y) *near* (a, b).

Slide 24

The tangent plane to the graph of f at a local max-min is horizontal

Theorem 6 *Let* $f(x, y)$ *be differentiable at* (a, b). *If* f *has a local maximum or minimum at* (a, b) *then* $\nabla f(a, b) = \langle 0, 0\rangle$.

Recall: $\mathbf{n} = \langle f_x, f_y, -1\rangle = \langle 0, 0, -1\rangle$.

The converse is not true: It could be a saddle point

Slide 25

Stationary points include local maxima, minima, and saddle points

Definition 6 (Stationary point) *Let $f(x, y)$ be a differentiable function at (a, b). If $\nabla f(a, b) = \langle 0, 0\rangle$, then the point (a, b) is called a stationary point of f.*

Stationary point are located where the gradient vector vanishes

Slide 26

Theorem 7 (Second derivative test) *Let (a, b) be a stationary point of $f(x, y)$, that is, $\nabla f(a, b) = \mathbf{0}$. Assume that $f(x, y)$ has continuous second derivatives in a disk with center in (a, b). Introduce the quantity*

$$D = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 .$$

- *If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.*
- *If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.*
- *If $D < 0$, then $f(a, b)$ is a saddle point.*
- *If $D = 0$ the test is inconclusive.*

Slide 27

Find the local extrema of $f(x,y) = y^2 - x^2$

$$\nabla f = \langle -2x, 2y \rangle, \quad \Rightarrow \quad \nabla f = \langle 0, 0 \rangle \quad \text{at} \quad (0,0).$$

$$f_{xx}(0,0) = -2, \quad f_{yy}(0,0) = 2, \quad f_{xy}(0,0) = 0,$$

$$D = (-2)(2) = -4 < 0 \quad \Rightarrow \quad \text{saddle point at } (0,0).$$

Slide 28

Is (0,0) a local extrema of $f(x,y) = y^2x^2$**?**

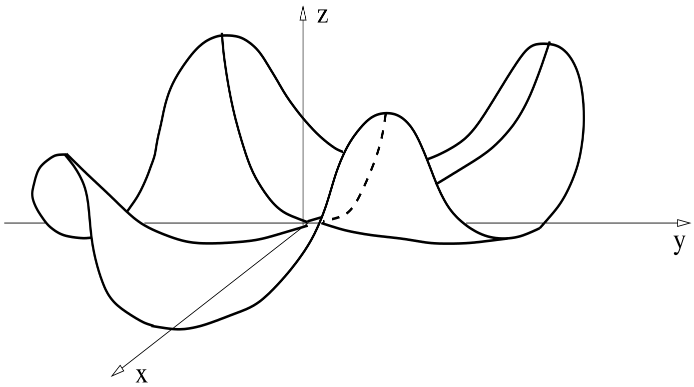
$$\nabla f(x,y) = \langle 2xy^2, 2yx^2 \rangle, \quad \Rightarrow$$
$$\nabla f(0,0) = \langle 0, 0 \rangle \quad \text{at} \quad (0,0).$$

$$f_{xx}(x,y) = 2y^2, \quad f_{yy}(x,y) = 2x^2, \quad f_{xy}(x,y) = 4xy,$$
$$f_{xx}(0,0) = 0, \quad f_{yy}(0,0) = 0, \quad f_{xy}(0,0) = 0,$$

So $D = 0$ and the test is inconclusive.

Slide 29

From the graph of $f = x^2y^2$ is easy to see that $(0,0)$ is a global minimum



Slide 30

Find the maximum volume of a closed rectangular box with a given surface area A_0

$$V(x,y,z) = xyz, \quad A(x,y,z) = 2xy + 2xz + 2yz.$$

But $A(x,y,z) = A_0$, then

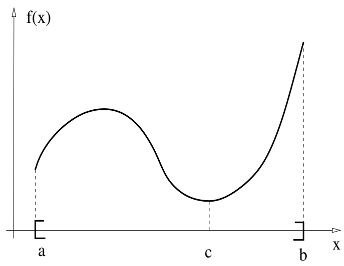
$$z = \frac{A_0 - 2xy}{2(x+y)}, \quad \Rightarrow \quad V(x,y) = \frac{A_0xy - 2x^2y^2}{2(x+y)}.$$

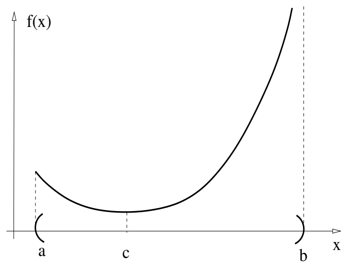
Find $\nabla V(x_0,y_0) = \langle 0,0 \rangle$.

The result is $x_0 = y_0 = z_0 = \sqrt{A_0/6}$.

Slide 31

Local extrema need not be the absolute extrema

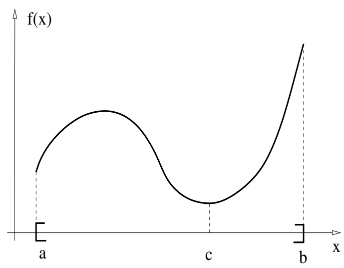




Absolute extrema may not be defined on open intervals

Slide 32

Continuous functions $f(x)$ on intervals $[a, b]$ always have absolute extrema



Intervals $[a, b]$ are bounded and closed sets in $\mathbb{R}$

Because they do not extend to infinity, and the boundary points belong to the set.

Slide 33

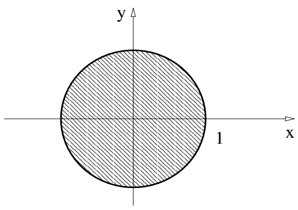
Here is the generalization of closed and bounded intervals to $\mathbb{R}^2$

Definition 7 *A set $D \subset \mathbb{R}^2$ is bounded if it can be contained in a disk. A set $D \in \mathbb{R}^2$ is closed if it contains all its boundary points.*

A point $P \in \mathbb{R}^2$ is a boundary point of a set D if every disk with center in P always contains both points in D and points not in D.

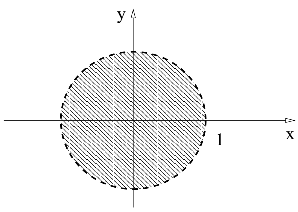
Slide 34

Here are examples of bounded sets



$\{x^2 + y^2 \leq 1\}$,

Closed and bounded.

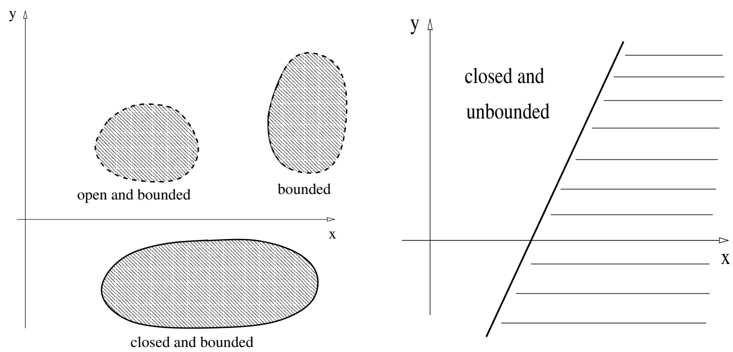


$\{x^2 + y^2 < 1\}$,

Open and bounded.

Slide 35

Here are more examples of different types of sets



Slide 36

Continuous functions on bounded and closed sets always have absolute extrema

Theorem 8 *If $f(x, y)$ is continuous in a closed and bounded set $D \subset \mathbb{R}^2$, then f has an absolute maximum and an absolute minimum in D.*

Slide 37

Suggestions to find absolute extrema of $f(x, y)$ in a closed and bounded set

- Find every stationary point of f.
  ($\nabla f(x, y) = \mathbf{0}$. No second derivative test needed.)
- Find the extrema (max. and min.) values of f on the boundary of D.
- The biggest (smallest) of the previous steps is the absolute maximum (minimum).

Slide 38

Here is a typical exercise

Find the absolute extrema of $f(x, y) = 4x + 6y - x^2 - y^2$, on $D = \{(x, y) \in I\!R^2, \quad 0 \leq x \leq 4, \quad 0 \leq y \leq 5\}$

Absolute minimum: $(4, 0)$, $(0, 0)$.
Absolute maximum: $(2, 3)$.

Slide 39

Lagrange's multipliers

- Example of the method.
- Maximization of functions subject to constraints.
- Examples.
- Generalization to more than one constraint.

Slide 40

Example: Find the rectangle of biggest area with fixed perimeter P_0

One way to solve the problem is:

$$A(x,y) = xy, \quad P_0 = P(x,y) = 2x + 2y,$$

then $y = P_0/2 - x$, and replace it in $A(x,y)$,

$$A(x) = \frac{P_0}{2}x - x^2.$$

The stationary points of this function are

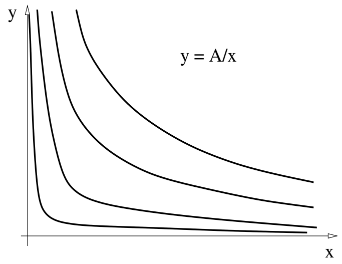
$$0 = A'(x) = \frac{P_0}{2} - 2x, \Rightarrow x = \frac{P_0}{4}, \Rightarrow y = \frac{P_0}{4}.$$

So the answer is the square of side

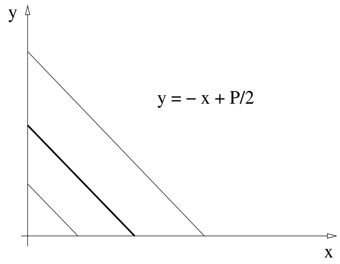
$$x = y = \frac{P_0}{4}.$$

Slide 41

Idea behind the Lagrange multipliers method



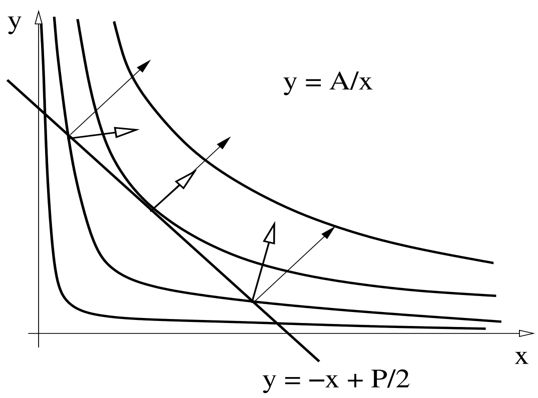
Level curves of $A = xy$,



Level curves of the constraint $P = 2x + 2y$.

Slide 42

The gradient vectors of $A(x, y)$ and of the constraint $P = 2x + 2y$ are parallel at the solution



Slide 43

The same problem solved with the Lagrange multipliers method

Find the maximum of $A(x, y) = xy$ *subject to the constraint* $P(x, y) = 2x + 2y = P_0$.

One has to find the (x, y) such that

$$\nabla A(x, y) = \lambda \nabla P(x, y), \quad P(x, y) = P_0,$$

with $\lambda \neq 0$. From the first equation one has

$$\langle y, x\rangle = \lambda\langle 2, 2\rangle, \quad \Rightarrow \quad x = 2\lambda, y = 2\lambda.$$

Then the constraint $P_0 = 2x + 2y$ implies that $P_0 = 8\lambda$, so the answer is

$$x = y = \frac{P_0}{4}.$$

Slide 44

Lagrange multipliers method can be summarized as follows:

The extrema values of $f(x, y)$ subject to the constraint $g(x, y) = k$ can be obtained as follows:

- Find all solutions (x_0, y_0) and λ of the equations

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0),$$
$$g(x_0, y_0) = k.$$

- Evaluate f at every solution (x_0, y_0). The largest and smallest values are respectively the maximum and minimum values of f subject to the constraint $g = k$.

Slide 45

Lagrange multipliers method for functions of three variables

The extrema values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ can be obtained as follows:

- Find all solutions (x_0, y_0, z_0) and λ of the equations

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0),$$
$$g(x_0, y_0, z_0) = k.$$

- Evaluate f at every solution (x_0, y_0, z_0). The largest and smallest values are respectively the maximum and minimum values of f subject to the constraint $g = k$.

Slide 46

Example: Find the rectangular box of maximum volume for fixed area.

The function is $V(x, y, z) = xyz$. The constraint function is $A(x, y, z) = 2xy + 2xz + 2yz$. The constraint is $A(x, y, z) = A_0$.

Find the (x, y, z) solutions of

$$\nabla V = \lambda \nabla A,$$
$$A = A_0.$$

These equations are:

$$yz = 2\lambda(z + y),$$
$$xz = 2\lambda(x + z),$$
$$xy = 2\lambda(x + y),$$
$$A_0 = 2(xy + xz + zy).$$

The solution is $x = y = z = \sqrt{A_0/6}$.

Slide 47

Example: Find the extrema values of $f(x,y) = x^2 + y^2/4$ in the circle $x^2 + y^2 = 1$

Then, $f(x,y) = x^2 + y^2/4$, and $g(x,y) = x^2 + y^2$. The equations are:

$$\nabla f = \lambda \nabla g, \quad \Rightarrow \quad \langle 2x, y/2 \rangle = \lambda \langle 2x, 2y \rangle,$$
$$g = 1, \quad \Rightarrow \quad x^2 + y^2 = 1.$$

Which imply

$$x = \lambda x, \quad \Rightarrow \quad (1-\lambda)x = 0,$$
$$y/2 = 2\lambda y, \quad \Rightarrow \quad (1/4 - \lambda)y = 0,$$
$$x^2 + y^2 = 1.$$

The solutions are: $P = (0, \pm 1)$, and $P = (\pm 1, 0)$. Then:

$f(0, \pm 1) = 1/4$, absolute minimum in the circle.

$f(\pm 1, 0) = 1$, absolute maximum in the circle.

Slide 48

Generalization to two constraints

The extrema values of $f(x,y,z)$ subject to the constraints $g(x,y,z) = k_1$ and $h(x,y,z) = k_2$ can be obtained as follows:

- Find all solutions (x_0, y_0, z_0) and λ of the equations

$$\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0) + \mu \nabla h(x_0,y_0,z_0),$$
$$g(x_0,y_0,z_0) = k_1,$$
$$h(x_0,y_0,z_0) = k_2.$$

- Evaluate f at every solution (x_0, y_0, z_0). The largest and smallest values are respectively the maximum and minimum values of f subject to the constraint $g = k_1$ and $h = k_2$.