Slide 1

Partial derivatives

- Review: Limits and continuity. (Sec. 14.2)
- Definition of Partial derivatives. (Sec. 14.3)
- Higher derivatives.
- Examples of differential equations.

Slide 2

We recall the definition of limit of $f(x, y)$

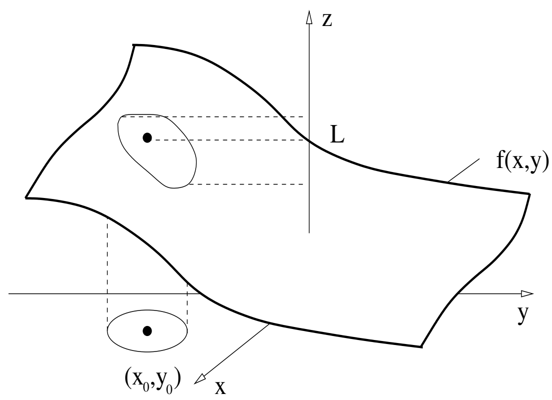
Let $f(x, y)$ be a scalar function defined for $P = (x, y)$ near $P_0 = (x_0, y_0)$. Let $d_{P_0P} = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ be the distance between (x, y) and (x_0, y_0). We write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L,$$

to mean that the values of $f(x, y)$ approaches L as the distance d_{P_0P} approaches zero.

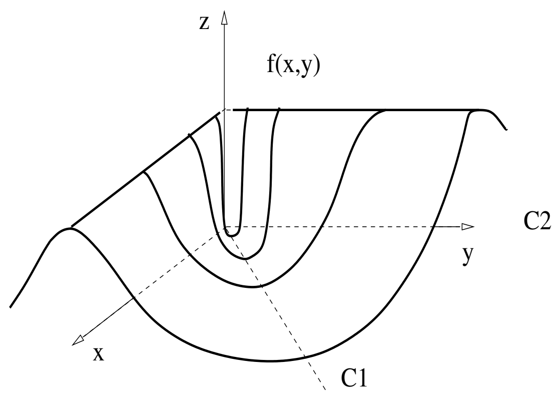
Slide 3

In this case $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ **exists**



Slide 4

In this case $\lim_{(x,y)\to(0,0)} f(x,y)$ **does not exist**



Compute side limits along C_1 **and** C_2

Slide 5

Continuous functions have graphs without holes or jumps

Definition 1 *$f(x, y)$ is continuous at (x_0, y_0) if*

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0).$$

Polynomial functions are continuous in $\mathbb{R}^2$, for example

$$P_2(x, y) = a_0 + b_1 x + b_2 y + c_1 x^2 + c_2 xy + c_3 y^2.$$

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More examples of continuous functions

- Rational functions are continuous on their domain,

  $$f(x, y) = \frac{P_n(x, y)}{Q_m(x, y)},$$

  for example,

  $$f(x, y) = \frac{x^2 + 3y - x^2y^2 + y^4}{x^2 - y^2}, \quad x \neq \pm y.$$

- Composition of continuous functions are continuous, example

  $$f(x, y) = \cos(x^2 + y^2).$$

Slide 7

To compute partial derivative with respect to x keep y constant

Definition 2 (x-partial derivative) *Let $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$. The partial derivative of $f(x, y)$ with respect to x at $(a, b) \in D$ is denoted as $f_x(a, b)$ and is given by*

$$f_x(a, b) = \lim_{h \to 0} \frac{1}{h} \left[f(a + h, b) - f(a, b) \right].$$

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To compute partial derivative with respect to y keep x constant

Definition 3 (y-partial derivative) *Let $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$. The partial derivative of $f(x, y)$ with respect to y at $(a, b) \in D$ is denoted as $f_y(a, b)$ and is given by*

$$f_y(a, b) = \lim_{h \to 0} \frac{1}{h} \left[f(a, b + h) - f(a, b) \right].$$

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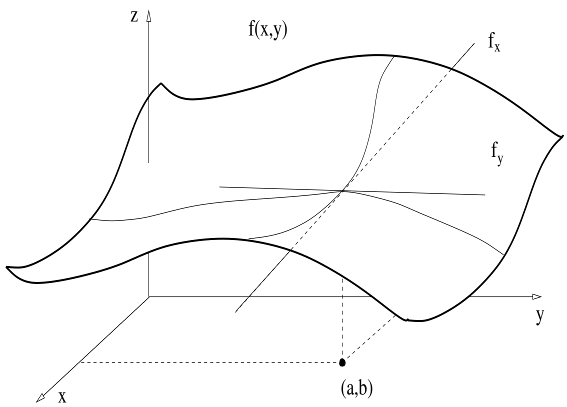
Partial derivatives are slopes of lines tangent to the graph of $f(x, y)$



So, to compute the partial derivative of $f(x, y)$ with respect to x at (a, b), one can do the following: First, evaluate the function at $y = b$, that is compute $f(x, b)$; second, compute the usual derivative of single variable functions; evaluate the result at $x = a$, and the result is $f_x(a, b)$.

Example:

- Find the partial derivative of $f(x, y) = x^2 + y^2/4$ with respect to x at $(1, 3)$.
  1. $f(x, 3) = x^2 + 9/4$;
  2. $f_x(x, 3) = 2x$;
  3. $f_x(1, 3) = 2$.

To compute the partial derivative of $f(x, y)$ with respect to y at (a, b), one follows the same idea: First, evaluate the function at $x = a$, that is compute $f(a, y)$; second, compute the usual derivative of single variable functions; evaluate the result at $y = b$,, and the result is $f_y(a, b)$.

Example:

- Find the partial derivative of $f(x, y) = x^2 + y^2/4$ with respect to y at $(1, 3)$.
  1. $f(1, y) = 1 + y^2/4$;
  2. $f_y(1, y) = y/2$;
  3. $f_y(1, 3) = 3/2$.

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Partial derivatives define new functions

Definition 4 *Consider a function* $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$. *The functions partial derivatives of* $f(x, y)$ *are denoted by* $f_x(x, y)$ *and* $f_y(x, y)$, *and are given by the expressions*

$$\begin{aligned} f_x(x, y) &= \lim_{h \to 0} \frac{1}{h} \left[f(x + h, y) - f(x, y) \right], \\ f_y(x, y) &= \lim_{h \to 0} \frac{1}{h} \left[f(x, y + h) - f(x, y) \right]. \end{aligned}$$

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The partial derivative functions of a paraboloid are planes

$$f(x, y) = ax^2 + by^2 + xy.$$

$$\begin{aligned} f_x(x, y) &= 2ax + 0 + y, \\ &= 2ax + y. \\ f_y(x, y) &= 0 + 2by + x, \\ &= 2by + x. \end{aligned}$$

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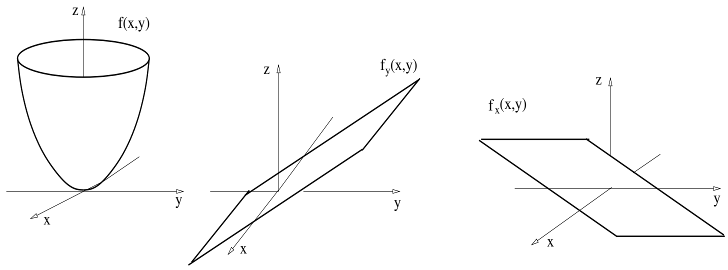
The partial derivative functions of a paraboloid are planes



More examples:

-
$$\begin{aligned} f(x,y) &= x^2\ln(y), \\ f_x(x,y) &= 2x\ln(y), \\ f_y(x,y) &= \frac{x^2}{y}. \end{aligned}$$

-
$$\begin{aligned} f(x,y) &= x^2 + \frac{y^2}{4}, \\ f_x(x,y) &= 2x, \\ f_y(x,y) &= \frac{y}{2}. \end{aligned}$$

-
$$\begin{aligned} f(x,y) &= \frac{2x-y}{x+2y}, \\ f_x(x,y) &= \frac{2(x+2y)-(2x-y)}{(x+2y)^2}, \\ &= \frac{2x+4y-2x+y}{(x+2y)^2}, \\ &= \frac{5y}{(x+2y)^2}. \end{aligned}$$

$$f_y(x,y) = \frac{-(x+2y) - (2x-y)2}{(x+2y)^2}, = \frac{-5x}{(x+2y)^2}.$$

- $$\begin{aligned} f(x,y) &= x^3 e^{2y} + 3y, \\ f_x(x,y) &= 3x^2 e^{2y}, \\ f_y(x,y) &= 2x^3 e^{2y} + 3, \\ f_{yy}(x,y) &= 4x^3 e^{2y}, \\ f_{yyy}(x,y) &= 8x^3 e^{2y}, \\ f_{xy} &= 6x^2 e^{2y}, \\ f_{yx} &= 6x^2 e^{2y}. \end{aligned}$$

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Higher derivatives of a function $f(x,y)$ are partial derivatives of its partial derivatives

For example, the second partial derivatives of $f(x,y)$ are the following:

$$f_{xx}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_x(x+h,y) - f_x(x,y)\right],$$
$$f_{yy}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_y(x,y+h) - f_y(x,y)\right],$$

$$f_{xy}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_x(x+h,y) - f_x(x,y)\right],$$
$$f_{yx}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_y(x,y+h) - f_y(x,y)\right].$$

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Higher partial derivatives sometimes commute

Theorem 1 *Consider a function $f(x, y)$ in a domain D. Assume that f_{xy} and f_{yx} exists and are continuous in D. Then,*

$$f_{xy} = f_{yx}.$$

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Differential equations are equations where the unknown is a function

For example, the Laplace equation: Find $\phi(x, y, z) : D \subset \mathbb{R}^3 \to \mathbb{R}$ solution of

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0.$$

This equation describes the gravitational effects near a massive object.

and where derivatives of the function enter into the equation

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More examples of differential equations

Heat equation: Find a function
$T(t, x, y, z) : D \subset \mathbb{R}^4 \to \mathbb{R}$ solution of

$$T_t = T_{xx} + T_{yy} + T_{zz}.$$

The heat on a metal is described by this equation. T is the temperature on that object.

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More examples of differential equations

Wave equation: Find a function
$f(t, x, y, z) : D \subset \mathbb{R}^4 \to \mathbb{R}$ solution of

$$f_{tt} = f_{xx} + f_{yy} + f_{zz}.$$

The sound in the air is described by this equation. f is the air density.

Exercises:

- Verify that the function $T(t,x) = e^{-t}\sin(x)$ satisfies the one-space dimensional heat equation $T_t = T_{xx}$.
- Verify that the function $f(t,x) = (t-x)^3$ satisfies the one-space dimensional wave equation $T_{tt} = T_{xx}$.
- Verify that the function below satisfies Laplace Equation,

$$\phi(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}.$$

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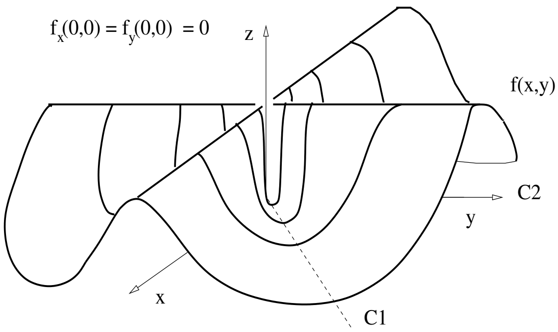
Differentiable functions (Sec. 14.4)

- Definition of differentiable functions.
- Equation of the tangent plane.
- Linear approximation. (Differentials.)

Slide 19

A function can have partial derivatives at a point and be discontinuous at that point



This is a very bad property for a definition of derivative

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Here is one of such functions, given explicitly

$$f(x,y) = \begin{cases} 2xy/(x^2+y^2) & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

$f_x(0,0) = f_y(0,0) = 0$, although $f(x,y)$ is not continuous at $(0,0)$.

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Recall the following property of the derivative of $f(x)$

Theorem 2 *If $f'(x)$ exists, then $f(x)$ is continuous.*

$$\begin{aligned}\lim_{h\to 0}[f(x+h)-f(x)] &= \lim_{h\to 0}\{[f(x+h)-f(x)]/h\}h,\\ &= \lim_{h\to 0} f'(x)h = 0.\end{aligned}$$

The analogous claim "If $f_x(x,y)$ and $f_y(x,y)$ exists, then $f(x,y)$ is continuous" is false

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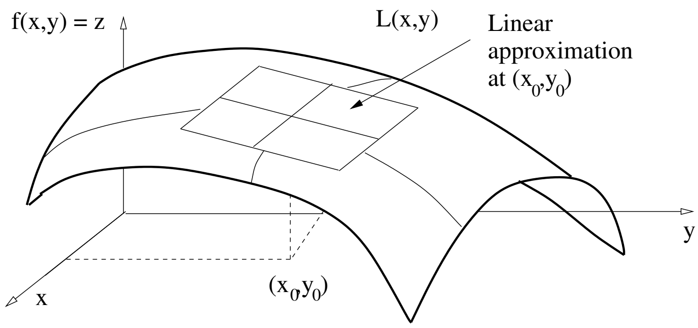
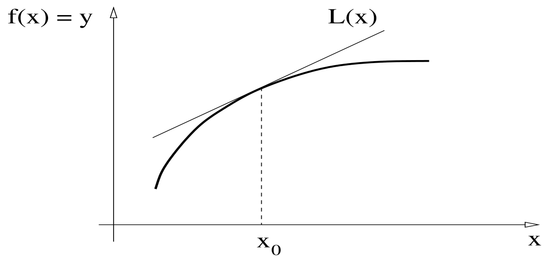
One has to define a notion of derivative having the continuity property discussed above



New definition: A differentiable function must be approximated by a plane

Slide 23

In the case $f(x)$ this definition says: The function must be approximated by a line



Only for functions $f(x)$ the derivative $f'(x)$ implies the existence of an approximating line $L(x)$

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A function of two variables is differentiable at (x_0, y_0) if two conditions hold:

- There exists the plane from its partial derivatives at (x_0, y_0);
- This plane approximates the graph of $f(x, y)$ near (x_0, y_0).

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Here is a rewording of the definition

Definition 5 *The function $f(x,y)$ is differentiable at (x_0,y_0) if*

$$f(x,y) = L_{(x_0,y_0)}(x,y) + \epsilon_1(x-x_0) + \epsilon_2(y-y_0),$$

where $\epsilon_i(x,y) \to 0$ when $(x,y) \to (x_0,y_0)$, for $i = 1,2$, and

$$L_{(x_0,y_0)}(x,y) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0)$$

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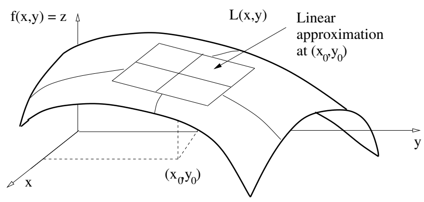
This notion of differentiability has the continuity property

Theorem 3 *If $f(x,y)$ is differentiable, then $f(x,y)$ is continuous.*



If $f(x,y)$ is differentiable, then $L_{(x_0,y_0)}(x,y)$ is called the linear approximation of $f(x,y)$ at (x_0,y_0).

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The following result is useful to check the differentiability of a function

Theorem 4 *Consider a function $f(x,y)$. Assume that its partial derivatives $f_x(x,y)$, $f_y(x,y)$ exist at (x_0,y_0) and near (x_0,y_0), and both are continuous functions at (x_0,y_0). Then, $f(x,y)$ is differentiable at (x_0,y_0).*

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Consider the following exercise:

1. Show that $f(x,y) = \arctan(x+2y)$ is differentiable at $(1,0)$.
2. Find its linear approximation at $(1,0)$.

$$f_x(x,y) = \frac{1}{1+(x+2y)^2}, \quad f_y(x,y) = \frac{2}{1+(x+2y)^2}.$$

These functions are continuous in $\mathbb{R}^2$, so $f(x,y)$ is differentiable at every point in $\mathbb{R}^2$.

$$L_{(1,0)}(x,y) = f_x(1,0)(x-1) + f_y(1,0)(y-0) + f(1,0),$$

where $f(1,0) = \arctan(1) = \pi/4$, $f_x(1,0) = 1/2$, $f_y(1,0) = 1$. Then,

$$L_{(1,0)}(x,y) = \frac{1}{2}(x-1) + y + \frac{\pi}{4}.$$

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Second exercise, on linear approximation

- Find the linear approximation of $f(x,y) = \sqrt{17 - x^2 - 4y^2}$ at $(2,1)$.

We need three numbers: $f(2,1)$, $f_x(2,1)$, and $f_y(2,1)$. Then, we compute the linear approximation by the formula

$$L_{(2,1)}(x,y) = f_x(2,1)(x-2) + f_y(2,1)(y-1) + f(2,1).$$

The result is: $f(2,1) = 3$, $f_x(2,1) = -2/3$, and $f_y(2,1) = -4/3$. Then the plane is given by

$$L_{(2,1)}(x,y) = -\frac{2}{3}(x-2) - \frac{4}{3}(y-1) + 3.$$

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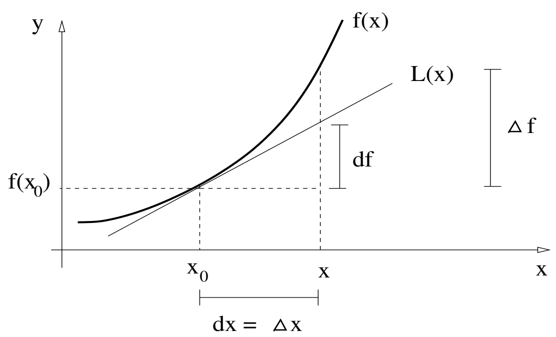
df is a special name for $L_{(x_0)}(x) - f(x_0)$

Single variable case:

$$df(x) = L_{x_0}(x) - f(x_0) = f'(x_0)(x - x_0) = f'(x_0)dx.$$

We called $(x - x_0) = dx$.



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$df(x,y)$ **is a special name for** $L_{(x_0,y_0)}(x,y) - f(x_0,y_0)$

Functions of two variables:

$$df(x,y) = L_{(x_0,y_0)}(x,y) - f(x_0,y_0),$$

$$dx = x - x_0, \quad dy = y - y_0.$$

Then, the formula is easy to remember:

$$df(x,y) = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy.$$

Different names for the same idea: Compute the linear approximation of a differentiable function.

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An exercise on differentials

- Compute the df of $f(x,y) = \ln(1 + x^2 + y^2)$ at $(1,1)$ for $dx = 0.1$, $dy = 0.2$.

$$\begin{aligned} df(x_0,y_0) &= f_x(x_0,y_0)dx + f_y(x_0,y_0)dy, \\ &= \frac{2x_0}{1+x_0^2+y_0^2}dx + \frac{2y_0}{1+x_0^2+y_0^2}dy. \end{aligned}$$

Then,

$$\begin{aligned} df(1,1) &= \frac{2}{3}\frac{1}{10} + \frac{2}{3}\frac{2}{10}, \\ &= \frac{2}{3}\frac{3}{10}, \\ &= \frac{1}{5}. \end{aligned}$$

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Another exercise on differentials

- Use differentials to estimate the amount of tin in a closed tin can with internal diameter of $8cm$ and height of $12cm$ if the tin is $0.04cm$ thick.

Data of the problem: $h_0 = 12cm$, $r_0 = 4cm$, $dr = 0.04cm$ and $dh = 0.08cm$. Draw a picture of the cylinder.

The function to consider is the volume of the cylinder,

$$V(r,h) = \pi r^2 h.$$

Then,

$$\begin{aligned} dV(r_0,h_0) &= V_r(r_0,h_0)dr + V_h(r_0,h_0)dh, \\ &= 2\pi r_0 h_0 dr + \pi r_0^2 dh \\ &= 16.1cm. \end{aligned}$$

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Chain rule and directional derivatives

- Review: Linear approximations. (Sec. 14.4)
- Chain rule. (Sec. 14.5)
- Cases 1 and 2. Examples.

Slide 35

Recall the chain rule for $f(x)$

Given $f(x)$, and $x(t)$ differentiable functions, introduce $z(t) = f(x(t))$. Then, $z(t)$ is differentiable, and

$$\frac{dz}{dt} = \frac{df}{dx}(x(t))\,\frac{dx}{dt}(t).$$

Or, using the new notation,

$$z_t(t) = f_x(x(t))\,x_t(t).$$

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There are many chain rules for $f(x, y)$

Case 1: Given $f(x, y)$ differentiable, and $x(t)$, $y(t)$ differentiable functions of a single variable, then $z(t) = f(x(t), y(t))$ is differentiable and

$$\frac{dz}{dt} = f_x(x(t), y(t))\,\frac{dx}{dt}(t) + f_y(x(t), y(t))\,\frac{dy}{dt}(t).$$

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Example of Chain rule, case 1

$f(x, y) = x^2 + 2y^3, \quad x(t) = \sin(t), \;\; y(t) = \cos(2t).$

Let $z(t) = f(x(t), y(t))$. Then,

$$\begin{aligned} \frac{dz}{dt} &= 2x(t)\frac{dx}{dt} + 6[y(t)]^2\frac{dy}{dt}, \\ &= 2x(t)\cos(t) - 12[y(t)]^2\sin(2t), \\ &= 2\sin(t)\cos(t) - 12\cos^2(2t)\sin(2t). \end{aligned}$$

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Second case of chain rule for $f(x, y)$

Case 2: Let $f(x, y)$ be differentiable, and $x(t, s)$, $y(t, s)$ be also differentiable functions of a two variable.

Then $z(t, s) = f(x(t, s), y(t, s))$ is differentiable and

$$z_t(t, s) = \\ f_x(x(t, s), y(t, s))\, x_t(t, s) + f_y(x(t, s), y(t, s))\, y_t(t, s)$$

$$z_s(t, s) = \\ f_x(x(t, s), y(t, s))\, x_s(t, s) + f_y(x(t, s), y(t, s))\, y_s(t, s)$$

Second case of chain rule for $f(x,y)$ again

Case 2: Let $f(x,y)$ be differentiable, and $x(t,s)$, $y(t,s)$ be also differentiable functions of a two variable.

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Then $z(t,s) = f(x(t,s), y(t,s))$ is differentiable and

$$z_t = f_x\, x_t + f_y\, y_t$$

$$z_s = f_x\, x_s + f_y\, y_s$$

Example of chain rule, case 2

A change of coordinates:

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Consider the function $f(x,y) = x^2 + ay^2$, with $a \in I\!R$. Introduce polar coordinates r, θ by the formula

$$x(r,\theta) = r\cos(\theta), \quad y(r,\theta) = r\sin(\theta).$$

Let $z(r,\theta) = f(x(r,\theta), y(r,\theta))$ be f in polar coordinates.

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A change of coordinates

Then, the chain rule, case 2, says that

$$z_r = f_x\, x_r + f_y\, y_r.$$

Each term can be computed as follows,

$$f_x = 2x, \quad f_y 2ay,$$

$$x_r = \cos(\theta), \quad y_r = \sin(\theta),$$

then one has

$$z_r = 2r\cos^2(\theta) + 2ar\sin^2(\theta).$$