Slide 1

Cross product and determinants

- Review: The dot product is a number.
- Geometric definition of cross product.
- Properties, and determinants.
- Cross product in components.
- Triple product and volumes.

Slide 2

The cross product of two vectors is another vector

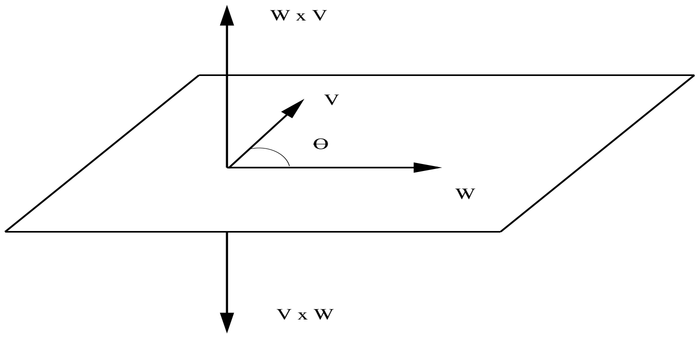
Definition 1 *Let* $\mathbf{v}$, $\mathbf{w}$ *be 3-dimensional vectors, and* $0 \leq \theta \leq \pi$ *be the angle in between them. Then,* $\mathbf{v} \times \mathbf{w}$ *is a vector perpendicular to* $\mathbf{v}$ *and* $\mathbf{w}$*, pointing in the direction given by the right hand rule, and with norm*

$$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}|\sin(\theta).$$

The cross product is perpendicular to the original vectors

Slide 3

The direction of the cross product vector is given by the right hand rule

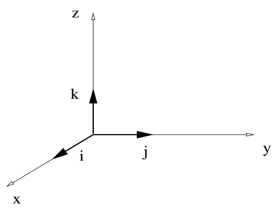


Tails together.

Slide 4

The cross products of the vectors i, j and k are simple to compute

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k}, & \mathbf{j} \times \mathbf{i} &= -\mathbf{k}, \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i}, & \mathbf{k} \times \mathbf{j} &= -\mathbf{i}, \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}, & \mathbf{i} \times \mathbf{k} &= -\mathbf{j}. \end{aligned}$$



Slide 5

Main properties of the cross product

- $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v} \quad \Rightarrow \quad \mathbf{v} \times \mathbf{v} = 0$,
- $(a\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a\mathbf{w}) = a(\mathbf{v} \times \mathbf{w})$,
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$,
- $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

Slide 6

The cross product is not associative

That is, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Example: $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = -\mathbf{k}$; but $(\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = 0$.

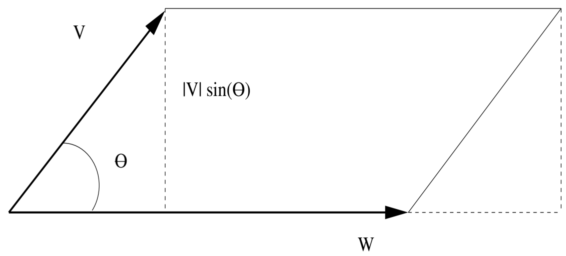
The cross product of two vectors vanishes when the vectors are parallel

Theorem 1 $\mathbf{v}$, $\mathbf{w} \neq 0$ *and* $\mathbf{v} \times \mathbf{w} = 0 \Leftrightarrow \mathbf{v}$ *parallel* $\mathbf{w}$.

Slide 7

The length of a cross product vector is an area

Theorem 2 $|\mathbf{v} \times \mathbf{w}|$ *is the area of the parallelogram formed by* $\mathbf{v}$ *and* $\mathbf{w}$*.*



Slide 8

The cross product can be written in terms of the components of the original vectors

Theorem 3 *Let* $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$*, and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$*. Then,*

$$\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.$$

For the proof of the last theorem, recall that

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Slide 9

We recall here the definition of determinant of a matrix

We use determinants only as a tool to remember the components of $\mathbf{v} \times \mathbf{w}$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Slide 10

The triple product of three vectors is a number

Definition 2 *Given* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, *the triple product is the number given by*

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

First do the cross product, and then dot the resulting vector with the third vector

Slide 11

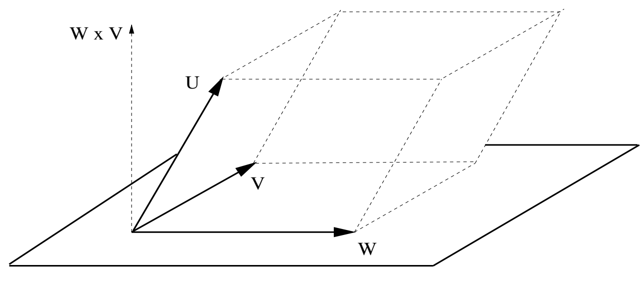
The triple product represents a volume of a solid formed with three vectors

Theorem 4 *Fix nonzero vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. *Then,* $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ *is the volume of the parallelepiped determined by* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

Notice that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v}$.

Slide 12

The triple product represents a volume of a solid formed with three vectors u, v, and w



Slide 13

Lines and planes in space

- Equations of lines.
  (Vector, parametric, and symmetric equations.)
- Equations of planes.
- Distance from a point to a plane.
  (Sec. 12.5)

Slide 14

A line is specified by a point and a tangent vector

Definition 3 *Let P_0 be a point in space, and $\mathbf{v}$ be a nonzero vector. Fix a coordinate system with origin at O, and let $\mathbf{r}_0 = \vec{OP}_0$. Then, the set of vectors*

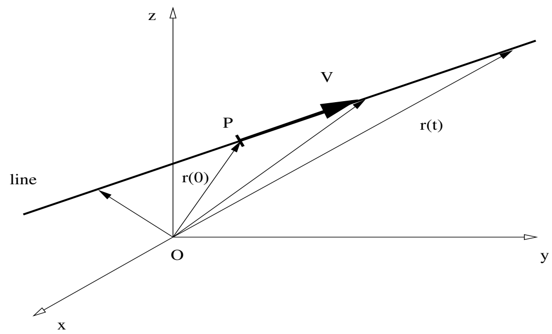
$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad t \in \mathbb{R},$$

is called the line through P_0 parallel to $\mathbf{v}$.

This is the vector equation of the line.

Slide 15

A line is specified by a point and a tangent vector



Slide 16

The parametric equations of a line is to have the functions $x(t)$, $y(t)$ and $z(t)$

Consider the case of 3 dimensions. In components,

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle,$$
$$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle,$$
$$\mathbf{v} = \langle v_x, v_y, v_z \rangle,$$

then the parametric equations of the line are

$$x(t) = x_0 + t v_x,$$
$$y(t) = y_0 + t v_y,$$
$$z(t) = z_0 + t v_z.$$

Slide 17

In the symmetric form of the equations the parameter t is taken out of the equations

Compute t in expressions above, and denote $x = x(t)$, $y = y(t)$, and $z = z(t)$. Then,

$$t = \frac{x - x_0}{v_x} = \frac{y - y_0}{v_y} = \frac{z - z_0}{v_z}.$$

These are called the symmetric equations of the line.

Slide 18

Two lines are parallel if their tangent vectors are parallel

Definition 4 *Two lines* $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$, *and* $\tilde{\mathbf{r}}(t) = \tilde{\mathbf{r}}_0 + t\tilde{\mathbf{v}}$ *are parallel if and only if* $\mathbf{v} = a\tilde{\mathbf{v}}$, *with* $a \neq 0$.

In $\mathbb{R}^2$ two lines are either parallel or they intersect (or both, when they coincide). This is not true in 3 dimensions.

Two lines in 3 dimensions are called skew lines if they are neither parallel nor they intersect.

Slide 19

A plane in space is determined with a point and a vector perpendicular to the plane

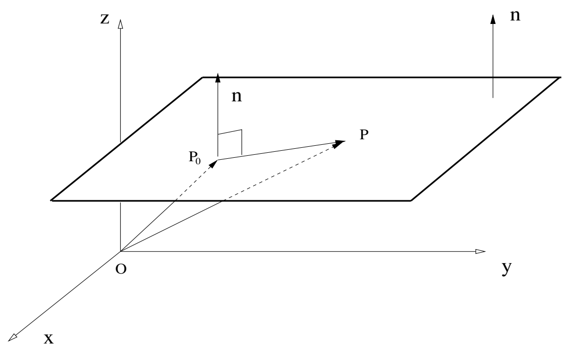
Definition 5 *Fix a point in space, P_0, and a nonzero vector* $\mathbf{n}$. *The set of all points P satisfying*

$$\overrightarrow{P_0P} \cdot \mathbf{n} = 0$$

is called the plane passing through P_0 normal to $\mathbf{n}$, *and we denote it as* $(P_0, \mathbf{n})$.

Slide 20

A plane in space is determined with a point and a vector perpendicular to the plane



Slide 21

In the equation of a plane appears the dot product

Theorem 5 *Fix a coordinate system with origin at O. The equation of the plane passing through P_0 normal to $\mathbf{n}$ can be written as*

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0,$$

where $\mathbf{r}_0 = \overrightarrow{OP}_0$, and $\mathbf{r} = \overrightarrow{OP}$, with P in the plane.

(Proof: $\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r}_0$.)

Slide 22

Here is the equations of a plane written explicitly in terms of coordinates

Theorem 6 *In components, the equation of a plane $(P_0, \mathbf{n})$ has the form*

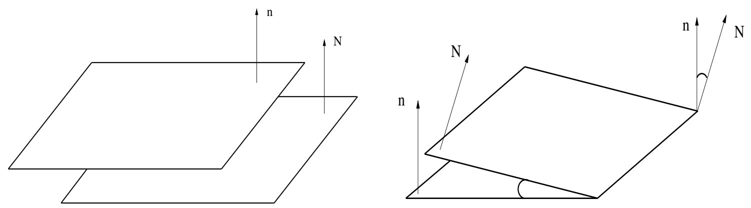
$$(x - x_0)n_x + (y - y_0)n_y + (z - z_0)n_z = 0,$$

where

$$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle, \quad \mathbf{n} = \langle n_x, n_y, n_z \rangle, \quad \mathbf{r} = \langle x, y, z \rangle.$$

Slide 23

Two planes are parallel if their normals are parallel



The angle between two non-parallel planes is the angle between their normal vectors

Slide 24

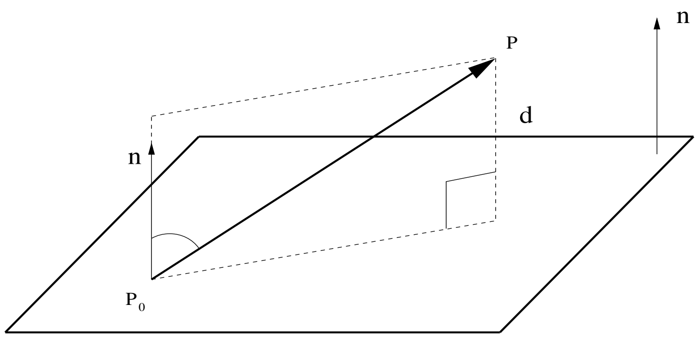
Distance formula from a point to a plane

Theorem 7 *The distance d from a point P_1 to a plane $(P_0, \mathbf{n})$ is the shortest distance from P_1 to any point in the plane, and is given by the expression*

$$d = \frac{|\vec{P_0 P} \cdot \mathbf{n}|}{|\mathbf{n}|}. \tag{1}$$

Slide 25

Distance from a point to a plane



Proof: Draw a picture of the situation. Then,

$$d = |\vec{P_0P}| \cos(\theta),$$

where θ is the angle between $\vec{P_0P}$ and $\mathbf{n}$. Recalling that

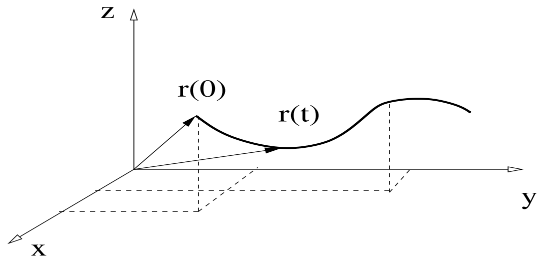
$$\vec{P_0P} \cdot \mathbf{n} = |\mathbf{n}|\,|\vec{P_0P}| \cos(\theta),$$

and that the distance is a nonnegative number, one gets Eq. (1).)

Slide 26

Vector valued functions, $\mathbf{r}(t)$

- Definition: $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$.
- Limits and derivatives.



Slide 27

Vector valued functions are 3 usual functions

Definition 6 *A vector valued function $\mathbf{r}(t)$ on $\mathbb{R}^3$ is a function*

$$\mathbf{r} : D \subset \mathbb{R} \to R \subset \mathbb{R}^3,$$

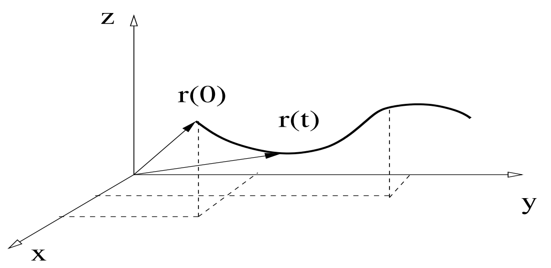
with $n \geq 2$, the set D is called domain of $\mathbf{r}$, and R is the range of $\mathbf{r}$.

In components, $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, where $x(t)$, $y(t)$ and $z(t)$ are usual functions

Slide 28

There is a natural association between curves in $\mathbb{R}^n$ and vector valued functions

The curve is determined by the head points of the vector valued function $\mathbf{r}(t)$.



The independent variable t is called the parameter of the curve.

Slide 29

The limit of $\mathbf{r}(t)$ as $t \to t_0$ is the limit of its components $x(t)$, $y(t)$, $z(t)$

Definition 7 *Consider the function*

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle.$$

The limit $\lim_{t\to t_0} \mathbf{r}(t)$ *is defined as*

$$\lim_{t\to t_0} \mathbf{r}(t) = \left\langle \lim_{t\to t_0} x(t), \lim_{t\to t_0} y(t), \lim_{t\to t_0} z(t) \right\rangle,$$

when such limit in each component exists.

Slide 30

Continuous vector valued functions

Definition 8 *A function* $\mathbf{r}(t)$ *is continuous at* t_0 *if*

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0).$$

Having the idea of limit, one can introduce the idea of a derivative of a vector valued function.

Slide 31

The derivative of a vector valued function $\mathbf{r}(t)$ **is another vector valued function,** $\mathbf{r}'(t)$

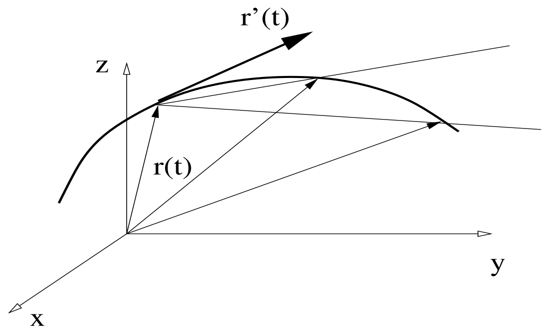
Definition 9 *The derivative of a vector valued function* $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$*, denoted a* $\mathbf{r}'(t)$*, is given by*

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h},$$

when such limit exists.

Slide 32

$\mathbf{r}'(t)$ is a vector tangent to the curve given by $\mathbf{r}(t)$



If $\mathbf{r}(t)$ represents the vector position of a particle, then $\mathbf{r}'(t)$ represents the velocity vector of that particle. That is, $\mathbf{v}(t) = \mathbf{r}'(t)$.

Slide 33

The derivative of $\mathbf{r}(t)$ is the derivative of its components

Theorem 8 *Consider the function*

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle,$$

where $x(t)$, $y(t)$, and $z(t)$ are differentiable functions. Then

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.$$

Proof:

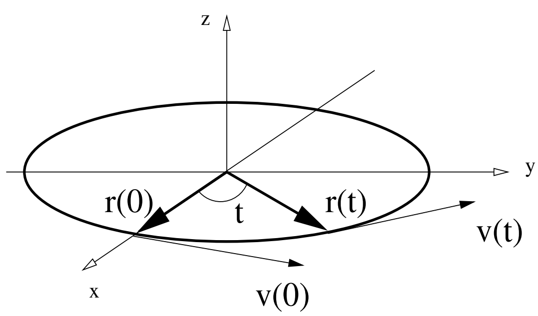
$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h},$$
$$= \lim_{h\to 0} \left\langle \frac{x(t+h)-x(t)}{h}, \frac{y(t+h)-y(t)}{h}, \frac{z(t+h)-z(t)}{h} \right\rangle$$
$$= \left\langle \lim_{h\to 0} \frac{x(t+h)-x(t)}{h}, \lim_{h\to 0} \frac{y(t+h)-y(t)}{h}, \lim_{h\to 0} \frac{z(t+h)-z(t)}{h} \right\rangle$$
$$= \langle x'(t), y'(t), z'(t)\rangle.$$

)

Slide 34

Examples of vector valued functions and their derivatives

$\mathbf{r}(t) = \langle \cos(t), \sin(t), 0\rangle \Rightarrow \mathbf{v}(t) = \langle -\sin(t), \cos(t), 0\rangle.$



Slide 35

Differentiation rules are the same as for usual (scalar) functions

- $[\mathbf{v}(t) + \mathbf{w}(t)]' = \mathbf{v}'(t) + \mathbf{w}'(t)$, (addition);
- $[c\mathbf{v}(t)]' = c\mathbf{v}'(t)$, (product rule for constants);
- $[f(t)\mathbf{v}(t)]' = f'(t)\mathbf{v}(t) + f(t)\mathbf{v}'(t)$, (product rule for scalar functions);
- $[\mathbf{v}(f(t))]' = \mathbf{v}'(f(t))f'(t)$, (chain rule for functions).

Slide 36

Differentiation rules for the new products on vectors

- $[\mathbf{v}(t) \cdot \mathbf{w}(t)]' = \mathbf{v}'(t) \cdot \mathbf{w}(t) + \mathbf{v}(t) \cdot \mathbf{w}'(t)$, (product rule for dot product);
- $[\mathbf{v}(t) \times \mathbf{w}(t)]' = \mathbf{v}'(t) \times \mathbf{w}(t) + \mathbf{v}(t) \times \mathbf{w}'(t)$, (product rule for cross product);

Slide 37

Higher derivatives can also be computed

The m derivative of $\mathbf{r}(t)$ is denoted as $\mathbf{r}^{(m)}(t)$ and is given by the expression $\mathbf{r}^{(m)}(t) = [\mathbf{r}^{(m-1)}(t)]'$.

Example:

$$\begin{aligned}
\mathbf{r}(t) &= \langle \cos(t), \sin(t), t^2 + 2t + 1 \rangle, \\
\mathbf{r}'(t) &= \langle -\sin(t), \cos(t), 2t + 2 \rangle, \\
\mathbf{r}^{(2)}(t) &= (\mathbf{r}'(t))' = \langle -\cos(t), -\sin(t), 2 \rangle, \\
\mathbf{r}^{(3)}(t) &= (\mathbf{r}^{(2)}(t))' = \langle \sin(t), -\cos(t), 0 \rangle.
\end{aligned}$$

If $\mathbf{r}(t)$ is the vector position of a particle, then the velocity vector is $\mathbf{v}(t) = \mathbf{r}'(t)$, and the acceleration vector is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}^{(2)}(t)$.

Slide 38

Definite integrals are computed component by component

Definition 10 *The definite integral of* $\mathbf{r}(t)$ *form* $t \in [a, b]$ *is a vector whose components are the integrals of the components of* $\mathbf{r}(t)$*, namely,*

$$\int_a^b \mathbf{r}(t)dt = \left\langle \int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt \right\rangle.$$

Slide 39

An explicit example

$$\begin{aligned}\int_0^{\pi} \langle \cos(t), \sin(t), t\rangle dt &= \left\langle \int_0^{\pi} \cos(t)dt, \int_0^{\pi} \sin(t)dt, \int_0^{\pi} tdt \right\rangle, \\ &= \left\langle \sin(t)|_0^{\pi}, -\cos(t)|_0^{\pi}, \left.\frac{t^2}{2}\right|_0^{\pi}, \right\rangle, \\ &= \left\langle 0, 2, \frac{\pi^2}{2} \right\rangle.\end{aligned}$$

Slide 40

How to compute derivatives in polar coordinates?

The definitions of limit and derivative of vector valued functions were introduced component by component in a fixed, given in advance, Cartesian coordinate system.

What does happen if one needs to work in other coordinate system, say, spherical coordinates, or arbitrary coordinates?

All the notions of limit and derivatives for vector valued functions can be generalized in a way that the Cartesian coordinates system need not to be introduced. When one introduce such Cartesian coordinates systems, one recovers the definitions presented here.

We do not study in our course this coordinate independent notion of limit and derivatives.