Slide 1

Coordinates in space

- Overview of vector calculus.
- Coordinate systems in space.
- Distance formula. (Sec. 12.1)

Slide 2

Vector calculus studies derivatives and integrals of functions of more than one variable

Math 20A studies: $f : \mathbb{R} \to \mathbb{R}$, $f(x)$, differential calculus.

Math 20B studies: $f : \mathbb{R} \to \mathbb{R}$, $f(x)$, integral calculus.

Math 20C considers:

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y);$$

$$f : \mathbb{R}^3 \to \mathbb{R}, \quad f(x, y, z);$$

$$\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, \quad \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle.$$

Slide 3

Incorporate one more axis to $I\!R^2$ and one gets $I\!R^3$

Every point in a plane can be labeled by an ordered pair of numbers, (x, y). (Descartes' idea.)

Every point in the space can be labeled by an ordered triple of numbers, (x, y, z).

There are two types of coordinates systems in space aside from rotations: *Right handed* and *Left handed.*

The same happens in $I\!R^2$.

Slide 4

The distance between points in space is crucial to generalize the idea of limit to functions in space

Theorem 1 *The distance between the points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is given by*

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Slide 5

A sphere is the set of points at fixed distance from a center

Application of the distance formula: The sphere centered at $P_0 = (x_0, y_0, z_0)$ of radius R are all points $P = (x, y, z)$ such that

$$|P_0P| = R,$$

that is,

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2.$$

Slide 6

Exercises with spheres

- Fix constants a, b, c, and d. Show that

  $$x^2 + y^2 + z^2 - 2ax - 2by - 2cz = d$$

  is the equation of a sphere if and only if

  $$d > -(a^2 + b^2 + c^2).$$

- Give the expressions for the center P_0 and the radius R of the sphere.

Slide 7

Vectors on the plane and in space

- Review: Distances and spheres.
- Vectors in R^2 and $I\!R^3$. (Sec 12.2)
- Components of a vector in a coordinate system.
- Operations: Geometric definition, and components.

The concept of vector is an abstraction that describes many different phenomena

$\sim$ 1800 Physicists and Mathematicians realized that several different physical phenomena were described using the same idea, the same concept. These phenomena included velocities, accelerations, forces, rotations, electric and magnetic phenomena, heat transfer, etc.

The new concept were more than a number in the sense that it was needed more than a single number to specify it.

Slide 8

A vector in $I\!R^2$ or in $I\!R^3$ is an oriented line segment

An oriented line segment has an initial (tail) point P_0 and a final (head) point P_1.

Notation: $\overrightarrow{P_0P_1}$, also , $\vec{v}$, and $\mathbf{v}$.

The length of a vector $\overrightarrow{P_0P_1}$ is denoted by $|\overrightarrow{P_0P_1}|$.

Slide 9

Vectors can be written in terms of components in a coordinate system

The vector with tail point $P_0 = (x_0, y_0, z_0)$ and head point $P_1 = (x_1, y_1, z_1)$ has components

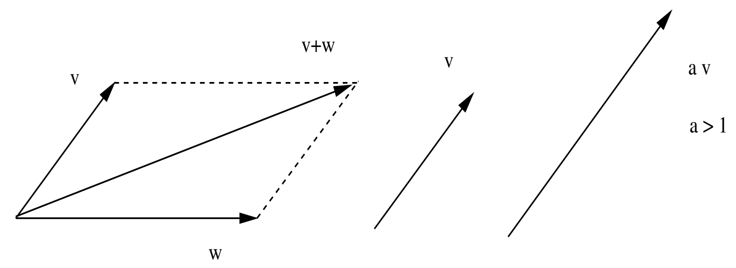
$$\overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle.$$

Points and a vector are different objects.

However, both are specified with an ordered pair of numbers in $I\!R^2$, or an ordered triple of numbers in $I\!R^3$.

Slide 10

The addition of two vectors is given by the parallelogram law



A vector can be stretched or compressed

Slide 11

The operations with vectors can be written in terms of components

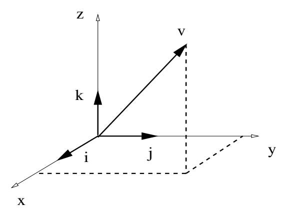
Given the vectors $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, $\mathbf{w} = \langle w_x, w_y, w_z \rangle$ in $I\!R^3$, and a number $a \in I\!R$, then the following expressions hold,

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= \langle (v_x + w_x), (v_y + w_y), (v_z + w_z) \rangle, \\ \mathbf{v} - \mathbf{w} &= \langle (v_x - w_x), (v_y - w_y), (v_z - w_z) \rangle, \\ a\mathbf{v} &= \langle av_x, av_y, av_z \rangle, \\ |\mathbf{v}| &= \Big[(v_x)^2 + (v_y)^2 + (v_z)^2 \Big]^{1/2}. \end{aligned}$$

Slide 12

The vectors i, j, k are very useful to write any other vector

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$



Every vector $\mathbf{v}$ in $\mathbb{R}^3$ can be written uniquely in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and the following equation holds, $\mathbf{v} = \langle v_x, v_y, v_z \rangle = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$.

Slide 13

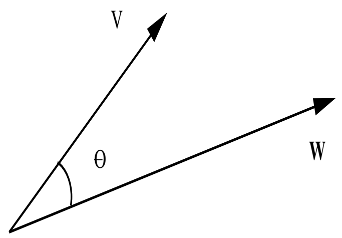
Dot product and projections

- Review: Parallelogram law and stretching.
- Dot product. Geometric definition.
- Orthogonal vectors, projections, and properties.
- Dot product in components.

Slide 14

The dot product of two vectors is a number

Definition 1 *Let* $\mathbf{v}$, $\mathbf{w}$ *be vectors and* $0 \le \theta \le \pi$ *be the angle in between. Then* $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta)$.

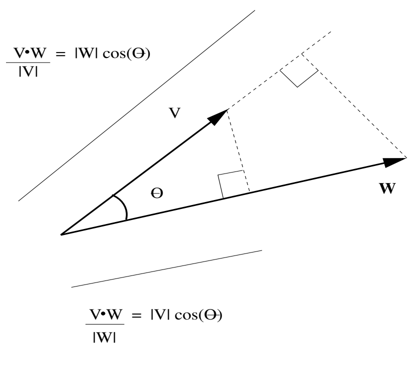


Tails together.

Slide 15

The dot product vanishes when the vectors are perpendicular

The dot product is closely related to projections of one vector onto the other.



Slide 16

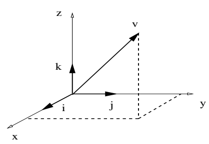
Here are some of the main properties of the dot product

- $\mathbf{v} \cdot \mathbf{w} = 0 \Longleftrightarrow \mathbf{v} \perp \mathbf{w}, \quad (\theta = \pi/2)$;
- $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2, \quad (\theta = 0)$;
- $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$, (commutative);
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, (distributive).

Slide 17

The dot product of the vectors i, j and k is very easy to compute

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle$$



$$\begin{aligned} \mathbf{i} \cdot \mathbf{i} &= 1, & \mathbf{j} \cdot \mathbf{j} &= 1, & \mathbf{k} \cdot \mathbf{k} &= 1, \\ \mathbf{i} \cdot \mathbf{j} &= 0, & \mathbf{j} \cdot \mathbf{i} &= 0, & \mathbf{k} \cdot \mathbf{i} &= 0, \\ \mathbf{i} \cdot \mathbf{k} &= 0, & \mathbf{j} \cdot \mathbf{k} &= 0, & \mathbf{k} \cdot \mathbf{j} &= 0 \end{aligned}$$

Slide 18

The dot product of two vectors can be written in terms of the components of the vectors

Theorem 2 *Let* $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, $\mathbf{w} = \langle w_x, w_y, w_z \rangle$. *Then*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z.$$

For the proof, recall that $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$, and $\mathbf{w} = w_x \mathbf{i} + w_y \mathbf{j} + w_z \mathbf{k}$, then the theorem follows from the distributive property of the dot product.