1. (a) (5 points) Find all constants $c$ such that the vectors $\mathbf{v} = \langle 1, c, 2 \rangle$ and $\mathbf{w} = \langle c^2, c, -4 \rangle$ are perpendicular to each other.

(b) (5 points) Set $c = 1$ in vectors $\mathbf{v}$ and $\mathbf{w}$ above. In this case, find a unit vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

(c) (5 points) Keep $c = 1$. Find the scalar projection of $\mathbf{v}$ onto $\mathbf{w}$.

(a)

$$0 = \mathbf{v} \cdot \mathbf{w} = \langle 1, c, 2 \rangle \cdot \langle c^2, c, -4 \rangle = c^2 + c^2 - 8 = 2c^2 - 8 \Rightarrow$$

$$\Rightarrow c^2 = 4 \Rightarrow c = \pm 2.$$ 

(b) $c = 1$ then $\mathbf{v} = \langle 1, 1, 2 \rangle$, $\mathbf{w} = \langle 1, 1, -4 \rangle$, then

$$\mathbf{u} = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 1 & 1 & -4 \end{vmatrix} = \langle -4 - 2, -(-4 - 2), (1 - 1) \rangle \Rightarrow$$

$$\Rightarrow \mathbf{u} = \langle -6, 6, 0 \rangle, \quad \Rightarrow |\mathbf{u}| = \sqrt{36 + 36} = 6\sqrt{2}.$$ 

Then a unit vector $\hat{\mathbf{u}}$ normal to both $\mathbf{v}$ and $\mathbf{w}$ is

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{\sqrt{2}} \langle -1, 1, 0 \rangle.$$ 

(c)

$$P_{\mathbf{v} \text{ onto } \mathbf{w}} = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|} = \frac{1 + 1 - 8}{\sqrt{1 + 1 + 16}} = \frac{-6}{\sqrt{18}} = -\frac{\sqrt{2}}{3}.$$
2. (10 points) Find the equation for the plane that contains the point \( P_0 = (1, 2, 3) \) and the line \( x = -2 + t, \ y = t, \ z = -1 + 2t \).

The equation of the line in vector form is 
\[
\mathbf{r}(t) = (-2, 0, -1) + (1, 1, 2)t
\]
so its tangent vector is \( \mathbf{v} = (1, 1, 2) \). The point \( P_0 = (1, 2, 3) \) is in the plane. A second point in the plane is any point in the line, for example \( P_1 \) corresponding to the head of \( \mathbf{r}(t = 0) = (-2, 0, -1) \). Then a second vector tangent to the plane is \( \overrightarrow{P_0 P_1} = (-3, -2, -4) \). Then, a normal to the plane is given by
\[
\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = ((4 - 4), -(4 - 6), (2 - 3)) \Rightarrow
\]
\[
\Rightarrow \quad \mathbf{n} = (0, 2, -1).
\]
So, the equation of the plane is
\[
0(x - 1) + 2(y - 2) - (z - 3) = 0, \quad \Rightarrow \quad 2y - z = 1.
\]
3. (a) (10 points) Find the position and velocity vector functions of a particle that moves with an acceleration function \( \mathbf{a}(t) = (0, 0, -10) \text{ m/sec}^2 \), knowing that the initial velocity and position are given by, respectively, \( \mathbf{v}(0) = (0, 1, 2) \text{ m/sec} \) and \( \mathbf{r}(0) = (0, 0, 3) \text{ m} \).

(b) (5 points) Draw an approximate picture of the graph of \( \mathbf{r}(t) \) for \( t \geq 0 \).

\[
\mathbf{a}(t) = (0, 0, -10),
\]

\[
\mathbf{v}(t) = (v_{0x}, v_{0y}, -10t + v_{0z}), \quad \mathbf{v}(0) = (0, 1, 2) \quad \Rightarrow \quad \begin{cases} v_{0x} = 0, \\ v_{0y} = 1, \\ v_{0z} = 2. \end{cases}
\]

\[
\mathbf{v}(t) = (0, 1, -10t + 2).
\]

\[
\mathbf{r}(t) = (r_{0x}, t + r_{0y}, -5t^2 + 2t + r_{0z}), \quad \mathbf{r}(0) = (0, 0, 3) \quad \Rightarrow \quad \begin{cases} r_{0x} = 0, \\ r_{0y} = 0, \\ r_{0z} = 3. \end{cases}
\]

\[
\mathbf{r}(t) = (0, t, -5t^2 + 2t + 3).
\]
4. (10 points) Reparametrize the curve \( \mathbf{r}(t) = \left\langle \frac{3}{2} \sin(t^2), 2t^2, \frac{3}{2} \cos(t^2) \right\rangle \) with respect to its arc length measured from \( t = 1 \) in the direction of increasing \( t \).

(Just in case you read it too fast, we repeat: starting at \( t = 1 \).)

\[
\mathbf{r}'(t) = (3t \cos(t^2), 4t, -3 \sin(t^2)),
\]
\[
|r'(t)| = \sqrt{9t^2 \cos^2(t^2) + 16t^2 + 9\sin^2(t^2)},
\]
\[
= \sqrt{9t^2 + 16t^2},
\]
\[
= \sqrt{9 + 16t},
\]
\[
= 5t.
\]

\[
s = \int_1^t 5t \, dt = \frac{5}{2} \left( t^2 \bigg|_1^t \right) = \frac{5}{2}(t^2 - 1).
\]
\[
t^2 = \frac{2}{5}s + 1.
\]

\[
\mathbf{r}(s) = \left\langle \frac{3}{2} \sin \left( \frac{2}{5}s + 1 \right), 2 \left( \frac{2}{5}s + 1 \right), \frac{3}{2} \cos \left( \frac{2}{5}s + 1 \right) \right\rangle.
\]