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Differentiable functions (Sec. 14.4)

- Review: Partial derivatives.
- Partial derivatives and continuity.
- Equation of the tangent plane.
- Differentiable functions.
- Application: Differentials. (Linear approximation.)

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Review: Partial derivatives

Definition 1 *Consider a function $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$. The functions partial derivatives of $f(x, y)$ are denoted by $f_x(x, y)$ and $f_y(x, y)$, and are given by the expressions*

$$f_x(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f(x+h,y) - f(x,y)\right],$$

$$f_y(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f(x,y+h) - f(x,y)\right].$$

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Review: Higher derivatives

Higher derivatives of a function $f(x, y)$ are partial derivatives of its partial derivatives. For example, the second partial derivatives of $f(x, y)$ are the following:

$$f_{xx}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_x(x+h,y) - f_x(x,y)\right],$$
$$f_{yy}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_y(x,y+h) - f_y(x,y)\right],$$

$$f_{xy}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_x(x+h,y) - f_x(x,y)\right],$$
$$f_{yx}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_y(x,y+h) - f_y(x,y)\right].$$

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Higher derivatives

Theorem 1 (Partial derivatives commute) *Consider a function $f(x, y)$ in a domain D. Assume that f_{xy} and f_{yx} exists and are continuous in D. Then,*

$$f_{xy} = f_{yx}.$$

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Examples of differential equations

Differential equations are equations where the unknown is a function, and where derivatives of the function enter into the equation. Examples:

- Laplace equation: Find $\phi(x, y, z) : D \subset I\!R^3 \to I\!R$ solution of

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0.$$

- Heat equation: Find a function $T(t, x, y, z) : D \subset I\!R^4 \to I\!R$ solution of

$$T_t = T_{xx} + T_{yy} + T_{zz}.$$

- Wave equation: Find a function $f(t, x, y, z) : D \subset I\!R^4 \to I\!R$ solution of

$$f_{tt} = f_{xx} + f_{yy} + f_{zz}.$$

Exercises:

- Verify that the function $T(t, x) = e^{-t} \sin(x)$ satisfies the one-space dimensional heat equation $T_t = T_{xx}$.
- Verify that the function $f(t, x) = (t - x)^3$ satisfies the one-space dimensional wave equation $T_{tt} = T_{xx}$.
- Verify that the function below satisfies Laplace Equation,

$$\phi(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

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Partial derivatives and continuity

Partial derivatives generalize the idea of derivative from single variable functions, $f(x)$ to functions $f(x, y)$, as follows,

Are the partial derivatives a faithful generalization?

NO.

Claim: If $f'(x)$ exists, then $f(x)$ is continuous.

True.

(Proof: $\lim_{h\to 0}[f(x+h) - f(x)] = \lim_{h\to 0}\{[f(x+h) - f(x)]/h\}h = \lim_{h\to 0} f'(x)h = 0$.)

Claim: If $f_x(x, y)$ and $f_y(x, y)$ exists, then $f(x, y)$ i continuous.

False.

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There is a counterexample:

$$f(x, y) = \begin{cases} 2xy/(x^2 + y^2) & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0). \end{cases}$$

What is a faithful generalization of the concept of derivative to functions $f(x, y)$?

The concept of linear approximation.

If $f'(x_0)$ exists, then $L(x) = f'(x_0)(x - x_0) + f(x_0)$ approximates $f(x)$ for x near x_0.

What is the analog of $L(x)$ in functions of two variables?

The analog to the line $L(x)$ is a plane $L(x, y)$.

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Summary

Consider a function $f(x, y)$ such that $f(x_0, y_0)$, $f_x(x_0, y_0)$, and $f_y(x_0, y_0)$ exist. Then, the plane

$$L_{(x_0,y_0)}(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$$

is well defined.

If this plane approximates $f(x, y)$ for (x, y) near (x_0, y_0), then we will say that $f(x, y)$ is differentiable at (x_0, y_0).

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Differentiable functions of two variables

Idea: A function $f(x, y)$ is differentiable at (x_0, y_0) if there exists the plane from its partial derivatives at (x_0, y_0),

AND

this plane approximates the graph of $f(x, y)$ near (x_0, y_0).

Definition 2 (Differentiable functions) *The function $f(x, y)$ is differentiable at (x_0, y_0) if*

$$f(x, y) = L_{(x_0,y_0)}(x, y) + \epsilon_1(x - x_0) + \epsilon_2(y - y_0),$$

and $\epsilon_i(x, y) \to 0$ when $(x, y) \to (x_0, y_0)$, for $i = 1, 2$.

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The following result is useful to check the differentiability of a function.

Theorem 2 *Consider a function $f(x, y)$. Assume that its partial derivatives $f_x(x, y)$, $f_y(x, y)$ exist at (x_0, y_0) and near (x_0, y_0), and both are continuous functions at (x_0, y_0).*

Then, $f(x, y)$ is differentiable at (x_0, y_0).

Definition 3 (Linear approximation) *If $f(x, y)$ is differentiable, then $L_{(x_0,y_0)}(x, y)$ is called the linear approximation of $f(x, y)$ at (x_0, y_0).*

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Differentials and chain rule

- Review: Differentiable functions. (Sec. 14.4)
- Linear approximation and differentials.
- Chain rule. (Sec. 14.5)

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Review: Differentiable functions

Let $f(x, y)$ be a function defined in a neighborhood of (x_0, y_0) such that the partial derivatives $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ exist.

Consider the plane $L_{(x_0,y_0)}(x, y)$ constructed with $f(x_0, y_0)$ and with the partial derivatives $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ given by

$$L_{(x_0,y_0)}(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0).$$

If this plane approximates the function $f(x, y)$ near (x_0, y_0), then we call $f(x, y)$ differentiable at (x_0, y_0).

(Then, for differentiable functions, the plane is called the linear approximation of $f(x, y)$ at (x_0, y_0).)

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Exercise: Differentiable functions

- Show that $f(x, y) = \arctan(x + 2y)$ is differentiable at $(1, 0)$.
- Find its linear approximation at $(1, 0)$.

$$f_x(x, y) = \frac{1}{1 + (x + 2y)^2}, \quad f_y(x, y) = \frac{2}{1 + (x + 2y)^2}.$$

These functions are continuous in $\mathbb{R}^2$, so $f(x, y)$ is differentiable at every point in $\mathbb{R}^2$.

$$L_{(1,0)}(x, y) = f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) + f(1, 0),$$

where $f(1, 0) = \arctan(1) = \pi/4$, $f_x(1, 0) = 1/2$, $f_y(1, 0) = 1$. Then,

$$L_{(1,0)}(x, y) = \frac{1}{2}(x - 1) + y + \frac{\pi}{4}.$$

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Exercise: Linear approximation

- Find the linear approximation of $f(x, y) = \sqrt{17 - x^2 - 4y^2}$ at $(2, 1)$.

We need three numbers: $f(2, 1)$, $f_x(2, 1)$, and $f_y(2, 1)$. Then, we compute the linear approximation by the formula

$$L_{(2,1)}(x, y) = f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) + f(2, 1).$$

The result is: $f(2, 1) = 3$, $f_x(2, 1) = -2/3$, and $f_y(2, 1) = -4/3$. Then the plane is given by

$$L_{(2,1)}(x, y) = -\frac{2}{3}(x - 2) - \frac{4}{3}(y - 1) + 3.$$

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Differentials

Different names for the same idea: Compute the linear approximation of a differentiable function.

The differential is a special name for $L_{(x_0,y_0)}(x, y) - f(x_0, y_0)$.

Single variable case:

$$df(x) = L_{x_0}(x) - f(x_0) = f'(x_0)(x - x_0) = f'(x_0)dx.$$

We called $(x - x_0) = dx$.

Functions of two variables:

$$df(x, y) = L_{(x_0,y_0)}(x, y) - f(x_0, y_0), \quad dx = x - x_0, \quad dy = y - y_0.$$

Then, the formula is easy to remember:

$$df(x, y) = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

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Exercise: Differentials

- Compute the df of $f(x,y) = \ln(1 + x^2 + y^2)$ at $(1,1)$ for $dx = 0.1$, $dy = 0.2$.

$$\begin{aligned} df(x_0,y_0) &= f_x(x_0,y_0)dx + f_y(x_0,y_0)dy, \\ &= \frac{2x_0}{1+x_0^2+y_0^2}dx + \frac{2y_0}{1+x_0^2+y_0^2}dy. \end{aligned}$$

Then,

$$\begin{aligned} df(1,1) &= \frac{2}{3}\frac{1}{10} + \frac{2}{3}\frac{2}{10}, \\ &= \frac{2}{3}\frac{3}{10}, \\ &= \frac{1}{5}. \end{aligned}$$

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Exercise: Differentials

- Use differentials to estimate the amount of tin in a closed tin can with internal diameter f $8cm$ and height of $12cm$ if the tin is $0.04cm$ thick.

Data of the problem: $h_0 = 12cm$, $r_0 = 4cm$, $dr = 0.04cm$ and $dh = 0.08cm$. Draw a picture of the cylinder.

The function to consider is the volume of the cylinder,

$$V(r,h) = \pi r^2 h.$$

Then,

$$\begin{aligned} dV(r_0,h_0) &= V_r(r_0,h_0)dr + V_h(r_0,h_0)dh, \\ &= 2\pi r_0 h_0 dr + \pi r_0^2 dh \\ &= 16.1cm. \end{aligned}$$

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Chain rule

- Single variable case. Given $f(x)$, and $x(t)$ differentiable functions, introduce $z(t) = f(x(t))$. Then, $z(t)$ is differentiable, and
$$\frac{dz}{dt} = \frac{df}{dx}(x(t))\,\frac{dx}{dt}(t).$$
Or, using the new notation,
$$z_t(t) = f_x(x(t))\,x_t(t).$$

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Chain rule

- Case 1: Given $f(x, y)$ differentiable, and $x(t)$, $y(t)$ differentiable functions of a single variable, then $z(t) = f(x(t), y(t))$ is differentiable and
$$\frac{dz}{dt} = f_x(x(t), y(t))\,\frac{dx}{dt}(t) + f_y(x(t), y(t))\,\frac{dy}{dt}(t).$$

Example: $f(x, y) = x^2 + 2y^3$, $x(t) = \sin(t)$, $y(t) = \cos(2t)$. Let $z(t) = f(x(t), y(t))$. Then,
$$\begin{aligned}\frac{dz}{dt} &= 2x(t)\frac{dx}{dt} + 6[y(t)]^2\frac{dy}{dt},\\ &= 2x(t)\cos(t) - 12[y(t)]^2\sin(2t),\\ &= 2\sin(t)\cos(t) - 12\cos^2(2t)\sin(2t).\end{aligned}$$

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Chain rule

- Case 2: Given $f(x, y)$ differentiable, and $x(t, s)$, $y(t, s)$ differentiable functions of a two variable, then $z(t, s) = f(x(t, s), y(t, s))$ is differentiable and

$$z_t(t,s) = f_x(x(t,s),y(t,s))\,x_t(t,s) + f_y(x(t,s),y(t,s))\,y_t(t,s),$$
$$z_s(t,s) = f_x(x(t,s),y(t,s))\,x_s(t,s) + f_y(x(t,s),y(t,s))\,y_s(t,s).$$

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Example: Change of coordinates

Consider the function $f(x, y) = x^2 + ay^2$, with $a \in \mathbb{R}$. Introduce polar coordinates r, θ by the formula

$$x(r, \theta) = r\cos(\theta), \quad y(r, \theta) = r\sin(\theta).$$

Let $z(r, \theta) = f(x(r, \theta), y(r, \theta))$. Then, the chain rule, case 2, says that

$$z_r = f_x x_r + f_y y_r.$$

Each term can be computed as follows,

$$f_x = 2x, \quad f_y 2ay,$$

$$x_r = \cos(\theta), \quad y_r = \sin(\theta),$$

then one has

$$z_r = 2r\cos^2(\theta) + 2ar\sin^2(\theta).$$