Slide 1

Limits and Continuity

- Review of Limit.
- Side limits and squeeze theorem.
- Continuous functions of 2,3 variables.

Slide 2

Review: Limits

Definition 1 *Given a function $f(x, y) : D \subset \mathbb{R}^2 \to \mathbb{R}$ and a point $(x_0, y_0) \in \mathbb{R}^2$, we write*

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L,$$

if and only if for all $(x, y) \in D$ close enough in distance to (x_0, y_0) the values of $f(x, y)$ approaches L.

Slide 3

A tool to show that a limit does not exist is the following:

Theorem 1 (Side limits) *If $f(x,y) \to L_1$ along a path C_1 as $(x,y) \to (x_0,y_0)$, and $f(x,y) \to L_2$ along a path C_2 as $(x,y) \to (x_0,y_0)$, with $L_1 \neq L_2$, then*

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \quad \textit{does not exist.}$$

A tool to prove that a limit exists is the following:

Theorem 2 (Squeeze) *Assume $f(x,y) \le g(x,y) \le h(x,y)$ for all (x,y) near (x_0,y_0);*
Assume

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L = \lim_{(x,y)\to(x_0,y_0)} h(x,y),$$

Then

$$\lim_{(x,y)\to(x_0,y_0)} g(x,y) = L.$$

Example: How to use the side limit theorem.

- Does the following limit exist?

$$\lim_{(x,y)\to(0,0)} \frac{3x^2}{x^2+2y^2}. \tag{1}$$

So, the function is $f(x,y) = (3x^2)/(x^2+2y^2)$. Let pick the curve C_1 as the x-axis, that is, $y=0$. Then,

$$f(x,0) = \frac{3x^2}{x^2} = 3,$$

then

$$\lim_{(x,0)\to(0,0)} f(x,0) = 3.$$

Let us now pick up the curve C_2 as the y-axis, that is, $x=0$. Then,

$$f(0,y) = 0,$$

then

$$\lim_{(x,0)\to(0,0)} f(x,0) = 0.$$

Therefore, the limit in (1) does not exist.

Notice that in the above example one could compute the limit for arbitrary lines, that is, C_m given by $y = mx$, with m a constant. Then

$$f(x,mx) = \frac{3x^2}{x^2+2m^2x^2} = \frac{3}{1+2m^2},$$

so one has that

$$\lim_{(x,mx)\to(0,0)} f(x,mx) = \frac{3}{1+2m^2}$$

is different for each value of m.

Example: How to use the squeeze theorem.

- Does the following limit exist?

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}. \tag{2}$$

Let us first try the side limit theorem, to try to prove that the limit does not exist. Consider the curves C_m given by $y = mx$, with m a constant. Then

$$f(x,mx) = \frac{x^2 mx}{x^2+m^2x^2} = \frac{mx}{1+m^2},$$

so one has that

$$\lim_{(x,mx)\to(0,0)} f(x,mx) = 0, \quad \forall\, m \in \mathbb{R}.$$

Therefore, one cannot conclude that the limit does not exist. However, this argument does not prove that the limit actually exists.This can be done with the squeeze theorem.

First notice that

$$\frac{x^2}{x^2+y^2} \leq 1, \quad \forall\, (x,y) \in \mathbb{R}^2, (x,y) \neq (0,0).$$

(proof: $0 \leq y^2$, then $x^2 \leq (x^2+y^2)$.) Therefore, one has the inequality

$$-|y| \leq \frac{x^2 y}{x^2+y^2} \leq |y|, \quad \forall\, (x,y) \in \mathbb{R}^2, (x,y) \neq (0,0).$$

Then, one knows that $\lim_{y\to 0} |y| = 0$, therefore the squeeze theorem says that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0.$$

Slide 4

Limits and Continuity

Limit laws for functions of a single variable also holds for functions of two variables.

If the limits $\lim_{\mathbf{x}\to\mathbf{x}_0} f$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} g$ exist, then

$$\lim_{\mathbf{x}\to\mathbf{x}_0} (f \pm g) = \left(\lim_{\mathbf{x}\to\mathbf{x}_0} f\right) \pm \left(\lim_{\mathbf{x}\to\mathbf{x}_0} g\right),$$

$$\lim_{\mathbf{x}\to\mathbf{x}_0} (fg) = \left(\lim_{\mathbf{x}\to\mathbf{x}_0} f\right) \left(\lim_{\mathbf{x}\to\mathbf{x}_0} g\right).$$

Definition 2 (Continuity) *A function $f(x,y)$ is continuous at (x_0,y_0) if*

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

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Continuity

Examples of continuous functions:

- Polynomial functions are continuous in $I\!R^2$, for example

$$P_2(x,y) = a_0 + b_1 x + b_2 y + c_1 x^2 + c_2 xy + c_3 y^2.$$

- Rational functions are continuous on their domain,

$$f(x,y) = \frac{P_n(x,y)}{Q_m(x,y)},$$

  for example,

$$f(x,y) = \frac{x^2 + 3y - x^2y^2 + y^4}{x^2 - y^2}, \quad x \neq \pm y.$$

- Composition of continuous functions are continuous, example

$$f(x,y) = \cos(x^2 + y^2).$$

Slide 6

Partial derivatives

- Definition of Partial derivatives.
- Higher derivatives.
- Examples of differential equations.

Slide 7

Partial derivatives

Definition 3 (Partial derivative) *Consider a function* $f : D \subset \mathbb{R}^2 \to R \subset \mathbb{R}$.

The partial derivative of $f(x, y)$ *with respect to* x *at* $(a, b) \in D$ *is denoted as* $f_x(a, b)$ *and is given by*

$$f_x(a,b) = \lim_{h\to 0} \frac{1}{h} \left[f(a+h,b) - f(a,b) \right].$$

The partial derivative of $f(x, y)$ *with respect to* y *at* $(a, b) \in D$ *is denoted as* $f_y(a, b)$ *and is given by*

$$f_y(a,b) = \lim_{h\to 0} \frac{1}{h} \left[f(a,b+h) - f(a,b) \right].$$

So, to compute the partial derivative of $f(x,y)$ with respect to x at (a,b), one can do the following: First, evaluate the function at $y=b$, that is compute $f(x,b)$; second, compute the usual derivative of single variable functions; evaluate the result at $x=a$, and the result is $f_x(a,b)$.

Example:

- Find the partial derivative of $f(x,y)=x^2+y^2/4$ with respect to x at $(1,3)$.
  1. $f(x,3)=x^2+9/4$;
  2. $f_x(x,3)=2x$;
  3. $f_x(1,3)=2$.

To compute the partial derivative of $f(x,y)$ with respect to y at (a,b), one follows the same idea: First, evaluate the function at $x=a$, that is compute $f(a,y)$; second, compute the usual derivative of single variable functions; evaluate the result at $y=b$,, and the result is $f_y(a,b)$.

Example:

- Find the partial derivative of $f(x,y)=x^2+y^2/4$ with respect to y at $(1,3)$.
  1. $f(1,y)=1+y^2/4$;
  2. $f_y(1,y)=y/2$;
  3. $f_y(1,3)=3/2$.

Discuss the geometrical meaning of the partial derivative using the graph of the function.

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Partial derivatives

The partial derivatives of f computed at any point $(x,y)\in D$ define the functions $f_x(x,y)$ and $f_y(x,y)$, the partial derivatives of f.

More precisely:

Definition 4 *Consider a function $f: D\subset \mathbb{R}^2 \to R\subset \mathbb{R}$. The functions partial derivatives of $f(x,y)$ are denoted by $f_x(x,y)$ and $f_y(x,y)$, and are given by the expressions*

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0}\frac{1}{h}\left[f(x+h,y)-f(x,y)\right], \\ f_y(x,y) &= \lim_{h\to 0}\frac{1}{h}\left[f(x,y+h)-f(x,y)\right]. \end{aligned}$$

Examples:

- $$\begin{aligned} f(x,y) &= ax^2 + by^2 + xy. \\ f_x(x,y) &= 2ax + 0 + y, \\ &= 2ax + y. \\ f_y(x,y) &= 0 + 2by + x, \\ &= 2by + x. \end{aligned}$$

- $$\begin{aligned} f(x,y) &= x^2 \ln(y), \\ f_x(x,y) &= 2x \ln(y), \\ f_y(x,y) &= \frac{x^2}{y}. \end{aligned}$$

- $$\begin{aligned} f(x,y) &= x^2 + \frac{y^2}{4}, \\ f_x(x,y) &= 2x, \\ f_y(x,y) &= \frac{y}{2}. \end{aligned}$$

- $$\begin{aligned} f(x,y) &= \frac{2x-y}{x+2y}, \\ f_x(x,y) &= \frac{2(x+2y)-(2x-y)}{(x+2y)^2}, \\ &= \frac{2x+4y-2x+y}{(x+2y)^2}, \\ &= \frac{5y}{(x+2y)^2}. \\ f_y(x,y) &= \frac{-(x+2y)-(2x-y)2}{(x+2y)^2}, \\ &= \frac{-5x}{(x+2y)^2}. \end{aligned}$$

- $$\begin{aligned} f(x,y) &= x^3 e^{2y} + 3y, \\ f_x(x,y) &= 3x^2 e^{2y}, \\ f_y(x,y) &= 2x^3 e^{2y} + 3, \\ f_{yy}(x,y) &= 4x^3 e^{2y}, \\ f_{yyy}(x,y) &= 8x^3 e^{2y}, \\ f_{xy} &= 6x^2 e^{2y}, \\ f_{yx} &= 6x^2 e^{2y}. \end{aligned}$$

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Higher derivatives

Higher derivatives of a function $f(x,y)$ are partial derivatives of its partial derivatives. For example, the second partial derivatives of $f(x,y)$ are the following:

$$f_{xx}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_x(x+h,y) - f_x(x,y)\right],$$
$$f_{yy}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_y(x,y+h) - f_y(x,y)\right],$$

$$f_{xy}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_x(x+h,y) - f_x(x,y)\right],$$
$$f_{yx}(x,y) = \lim_{h\to 0} \frac{1}{h}\left[f_y(x,y+h) - f_y(x,y)\right].$$

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Higher derivatives

Theorem 3 (Partial derivatives commute) *Consider a function $f(x, y)$ in a domain D. Assume that f_{xy} and f_{yx} exists and are continuous in D. Then,*

$$f_{xy} = f_{yx}.$$

Slide 11

Examples of differential equations

Differential equations are equations where the unknown is a function, and where derivatives of the function enter into the equation. Examples:

- Laplace equation: Find $\phi(x, y, z) : D \subset \mathbb{R}^3 \to \mathbb{R}$ solution of
$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0.$$
- Heat equation: Find a function $T(t, x, y, z) : D \subset \mathbb{R}^4 \to \mathbb{R}$ solution of
$$T_t = T_{xx} + T_{yy} + T_{zz}.$$
- Wave equation: Find a function $f(t, x, y, z) : D \subset \mathbb{R}^4 \to \mathbb{R}$ solution of
$$f_{tt} = f_{xx} + f_{yy} + f_{zz}.$$

Exercises:

- Verify that the function $T(t,x) = e^{-t}\sin(x)$ satisfies the one-space dimensional heat equation $T_t = T_{xx}$.
- Verify that the function $f(t,x) = (t-x)^3$ satisfies the one-space dimensional wave equation $T_{tt} = T_{xx}$.
- Verify that the function below satisfies Laplace Equation,

$$\phi(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}.$$