Slide 1

Coordinates in space

- Overview of vector calculus.
- Coordinate systems in space.
- Distance formula. (Sec. 12.1)

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Vector calculus studies derivatives and integrals of functions of more than one variable

Math 20A studies: $f : \mathbb{R} \to \mathbb{R}$, $f(x)$, differential calculus.

Math 20B studies: $f : \mathbb{R} \to \mathbb{R}$, $f(x)$, integral calculus.

Math 20C considers:

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y);$$

$$f : \mathbb{R}^3 \to \mathbb{R}, \quad f(x, y, z);$$

$$\mathbf{r} : \mathbb{R} \to \mathbb{R}^3, \quad \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle.$$

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Incorporate one more axis to $I\!R^2$ and one gets $I\!R^3$

Every point in a plane can be labeled by an ordered pair of numbers, (x, y). (Descartes' idea.)

Every point in the space can be labeled by an ordered triple of numbers, (x, y, z).

There are two types of coordinates systems in space aside from rotations: *Right handed* and *Left handed.*

The same happens in $I\!R^2$.

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The distance between points in space is crucial to generalize the idea of limit to functions in space

Theorem 1 *The distance between the points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is given by*

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

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A sphere is the set of points at fixed distance from a center

Application of the distance formula: The sphere centered at $P_0 = (x_0, y_0, z_0)$ of radius R are all points $P = (x, y, z)$ such that

$$|P_0P| = R,$$

that is,

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2.$$

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Exercises with spheres

- Fix constants a, b, c, and d. Show that

$$x^2 + y^2 + z^2 - 2ax - 2by - 2cz = d$$

  is the equation of a sphere if and only if

$$d > -(a^2 + b^2 + c^2).$$

- Give the expressions for the center P_0 and the radius R of the sphere.

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Vectors in space

- Definition and main operations:
  - Addition, Difference.
  - Multiplication by a number.
- Components of a vector in a coordinate system.

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What are vectors? $\sim$ 1800 Physicists and

Mathematicians realized that several different physical phenomena were described using the same idea, the same concept. These phenomena included velocities, accelerations, forces, rotations, electric and magnetic phenomena, heat transfer, etc.

The new concept were more than a number in the sense that it was needed more than a single number to specify it.

Definition 1 *A vector in* $\mathbb{R}^3$ *is an oriented segment.*

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Operations with vectors

- Addition: Parallelogram law.
- Multiplication by a number. (Positive, negative, or zero.)
- Difference.

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Components on a vector The operations with vectors, defined geometrically can be written in terms of components.

Given the vectors $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, $\mathbf{w} = \langle w_x, w_y, w_z \rangle$ in $\mathbb{R}^3$, and a number $a \in \mathbb{R}$, then the following expressions hold,

$$\begin{aligned}
\mathbf{v} + \mathbf{w} &= \langle (v_x + w_x), (v_y + w_y), (v_z + w_z) \rangle, \\
\mathbf{v} - \mathbf{w} &= \langle (v_x - w_x), (v_y - w_y), (v_z - w_z) \rangle, \\
a\mathbf{v} &= \langle a v_x, a v_y, a v_z \rangle, \\
|\mathbf{v}| &= \left[(v_x)^2 + (v_y)^2 + (v_z)^2 \right]^{1/2}.
\end{aligned}$$

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Useful vectors:

$$\begin{aligned} \mathbf{i} &= \langle 1, 0, 0 \rangle, \\ \mathbf{j} &= \langle 0, 1, 0 \rangle, \\ \mathbf{k} &= \langle 0, 0, 1 \rangle, \end{aligned}$$

Every vector $\mathbf{v}$ in $I\!R^3$ can be written uniquely in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

The following equation holds,

$$\mathbf{v} = \langle v_x, v_y, v_z \rangle = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}.$$

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Dot Product

- Definition
- Properties
- Equivalent expression

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Definition and properties

Definition 2 *Let* $\mathbf{v}$, $\mathbf{w}$ *be vectors and* $0 \leq \theta \leq \pi$ *be the angle in between. Then*

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos(\theta).$$

Properties:

- $\mathbf{v} \cdot \mathbf{w} = 0 \Longleftrightarrow \mathbf{v} \perp \mathbf{w}, \quad (\theta = \pi/2);$
- $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2, \quad (\theta = 0);$
- $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}, \quad$ (commutative);
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}.$

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Equivalent expression

Theorem 2 *Let* $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, $\mathbf{w} = \langle w_x, w_y, w_z \rangle$. *Then*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z.$$

For the proof, recall that

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle$$

$$\begin{aligned} \mathbf{i} \cdot \mathbf{i} &= 1, & \mathbf{j} \cdot \mathbf{j} &= 1, & \mathbf{k} \cdot \mathbf{k} &= 1, \\ \mathbf{i} \cdot \mathbf{j} &= 0, & \mathbf{j} \cdot \mathbf{i} &= 0, & \mathbf{k} \cdot \mathbf{i} &= 0, \\ \mathbf{i} \cdot \mathbf{k} &= 0, & \mathbf{j} \cdot \mathbf{k} &= 0, & \mathbf{k} \cdot \mathbf{j} &= 0 \end{aligned}$$

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Cross Product

- Definition
- Properties (Determinants)
- Equivalent expression
- Triple product

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Definition

Definition 3 *Let* $\mathbf{v}$, $\mathbf{w}$ *be 3-dimensional vectors, and* $0 \leq \theta \leq \pi$ *be the angle in between them. Then,* $\mathbf{v} \times \mathbf{w}$ *is a vector normal to* $\mathbf{v}$ *and* $\mathbf{w}$*, pointing in the direction given by the right hand rule, and with norm*

$$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}| \sin(\theta).$$

Example:

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k}, & \mathbf{j} \times \mathbf{i} &= -\mathbf{k}, \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i}, & \mathbf{k} \times \mathbf{j} &= -\mathbf{i}, \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j}, & \mathbf{i} \times \mathbf{k} &= -\mathbf{j}. \end{aligned}$$

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Properties

- $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$,
- $\mathbf{v} \times \mathbf{v} = 0$,
- $(a\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a\mathbf{w}) = a(\mathbf{v} \times \mathbf{w})$,
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$,
- $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

Notice: $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Example: $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = -\mathbf{k}$, but $(\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = 0$.

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Theorem 3 *If $\mathbf{v}$, $\mathbf{w} \neq 0$, then the following assertion holds:*

$$\mathbf{v} \times \mathbf{w} = 0 \Leftrightarrow \mathbf{v} \text{ parallel } \mathbf{w}.$$

Theorem 4 *$|\mathbf{v} \times \mathbf{w}|$ is the area of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$.*

Theorem 5 *Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$. Then,*

$$\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.$$

For the proof of the last theorem, recall that

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

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Note on determinants

They are useful un several areas of Mathematics. We don't study them in our course. We use them only as a tool to remember the components of $\mathbf{v} \times \mathbf{w}$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

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Triple product

Definition 4 *Given* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, *the triple product is the number given by*

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

Theorem 6 *Fix nonzero vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. *Then,* $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ *is the volume of the parallelepiped determined by* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

Note: $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v}$.