Print Name: ______________________________ **Section Number:** ______

TA Name: ______________________________ **Section Time:** ______

Math 20C.
Final Exam
June 15, 2006

No calculators or any other devices are allowed on this exam.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
Read each question carefully. If any question is not clear, ask for clarification.
Answer each question completely, and show all your work.

1. (10 points) Find the plane through the point $P_0 = (2, 1, -1)$ which is perpendicular to the planes $2x + y - z = 3$ and $x + 2y + z = 2$.

The plane is determined by its normal vector $\mathbf{n}$ and a point. We choose the point to be $P_0 = (2, 1, -1)$. The normal vector can be computed as

$$\mathbf{n} = \mathbf{n}_1 \times \mathbf{n}_2, \qquad \mathbf{n}_1 = \langle 2, 1, -1\rangle, \quad \mathbf{n}_1 = \langle 1, 2, 1\rangle.$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the normal vectors to the planes $2x + y - z = 3$ and $x + 2y + z = 2$, respectively. Then,

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \langle (1+2), -(2+1), (4-1)\rangle \quad \Rightarrow \quad \mathbf{n} = \langle 3, -3, 3\rangle.$$

We can pick up any vector proportional to $\langle 3, -3, 3\rangle$ as the normal vector to the plane, for example a simpler one is $\mathbf{n} = \langle 1, -1, 1\rangle$. Then, the equation of the plane is

$$(x-2) - (y-1) + (z+1) = 0 \quad \Rightarrow \quad \boxed{x - y + z = 0}.$$

2. (8 points) Decide whether the $\lim_{(x,y)\to(0,0)} \frac{x^4-y^2}{x^4+y^2}$ exists. Give reasons your answer.

Consider the path given by the line $x = 0$, then

$$\lim_{(0,y)\to(0,0)} \frac{x^4-y^2}{x^4+y^2} = \lim_{y\to 0} \frac{-y^2}{+y^2} = \lim_{y\to 0} -1 = -1.$$

Consider the path given by the line $y = 0$, then

$$\lim_{(x,0)\to(0,0)} \frac{x^4-y^2}{x^4+y^2} = \lim_{x\to 0} \frac{x^4}{x^4} = \lim_{x\to 0} 1 = 1.$$

Therefore, the limit does not exist.

3. (8 points) Does the function $f(x, y, z) = e^{3x+4y}\cos(5z)$ satisfy the Laplace equation $f_{xx} + f_{yy} + f_{zz} = 0$? Give reasons your answer.

$$\begin{aligned} f_x &= 3e^{3x+4y}\cos(5z), & f_y &= 4e^{3x+4y}\cos(5z), & f_z &= -5e^{3x+4y}\sin(5z) \\ f_{xx} &= 9e^{3x+4y}\cos(5z), & f_{yy} &= 16e^{3x+4y}\cos(5z), & f_{zz} &= -25e^{3x+4y}\cos(5z), \end{aligned}$$

therefore,

$$f_{xx} + f_{yy} + f_{zz} = (9 + 16 - 25)e^{3x+4y}\cos(5z) = 0 \quad \Rightarrow \quad \boxed{f_{xx} + f_{yy} + f_{zz} = 0}.$$

4. (10 points) Find the linear approximation $L(x,y)$ of the function $f(x,y) = \sqrt{6-x^2-y^2}$ at the point $(1,1)$. Use this approximation to estimate the value of $f(0.8, 1.1)$.

$$f(x,y) = \sqrt{6-x^2-y^2}, \qquad f(1,1) = \sqrt{6-2} = 2,$$

$$f_x(x,y) = \frac{-x}{\sqrt{6-x^2-y^2}}, \qquad f_x(1,1) = \frac{-1}{\sqrt{6-2}} = -\frac{1}{2},$$

$$f_y(x,y) = \frac{-y}{\sqrt{6-x^2-y^2}}, \qquad f_y(1,1) = \frac{-1}{\sqrt{6-2}} = -\frac{1}{2}.$$

$$\boxed{L(x,y) = -\frac{1}{2}(x-1) - \frac{1}{2}(y-1) + 2}.$$

$$L(0.8, 1.1) = -\frac{1}{2}(-0.2) - \frac{1}{2}(0.1) + 2 = \frac{1}{2}(0.1) + 2 = 2 + \frac{1}{20} = \frac{41}{20}.$$

$$\boxed{L(0.8, 1.1) = \frac{41}{20}}.$$

5. (10 points) Find the local maxima, local minima and saddle points of the function $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$.

$$\nabla f = \langle 3x^2 + 6x, 3y^2 - 6y \rangle = \langle 0, 0 \rangle \quad \Rightarrow \quad \begin{cases} 3x(x+2) = 0 \\ 3y(y-2) = 0 \end{cases}$$

so $x = 0$ or $x = -2$, while $y = 0$ or $y = 2$. Then, there are four stationary points given by

$$(0,0), \quad (0,2), \quad (-2,0), \quad (-2,2).$$

$$f_{xx} = 6x + 6, \qquad f_{yy} = 6y - 6, \qquad f_{xy} = 0.$$

Therefore,

$$D = f_{xx} f_{yy} - \left(f_{xy}\right)^2 = f_{xx} f_{yy} = 36(x+1)(y-1).$$

Evaluating it at each stationary point we get:

$$\begin{aligned} D(0,0) &= -36, & & & &\boxed{(0,0) \text{ saddle point}}, \\ D(0,2) &= 36, & f_{xx}(0,2) &= 6, & &\boxed{(0,2) \text{ local minimum}}, \\ D(-2,0) &= 36, & f_{xx}(-2,0) &= -6, & &\boxed{(0,2) \text{ local maximum}}, \\ D(-2,2) &= -36, & & & &\boxed{(-2,2) \text{ saddle point}}. \end{aligned}$$

6. (10 points) Use Lagrange multipliers to find the maximum and minimum values of the function $f(x,y) = -\dfrac{1}{x} + \dfrac{1}{y}$ subject to the constraint $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 1$.

Denote $g(x,y) = \frac{1}{x^2} + \frac{1}{y^2} - 1$, so the constraint is $g = 0$. The Lagrange multipliers equations are

$$\nabla f = \lambda \nabla g, \quad \text{and} \quad g = 0.$$

$$\left\langle \frac{1}{x^2}, -\frac{1}{y^2} \right\rangle = \lambda \left\langle -\frac{2}{x^3}, -\frac{2}{y^3} \right\rangle \quad \Rightarrow \quad \begin{cases} \dfrac{1}{x^2} = -\dfrac{2\lambda}{x^3}, \\ -\dfrac{1}{y^2} = -\dfrac{2\lambda}{y^3}. \end{cases}$$

Then x and y must be nonzero, so,

$$x = -2\lambda, \quad y = 2\lambda \quad \Rightarrow \quad x = -y.$$

Then, using this information in the constraint we have

$$\frac{1}{x^2} + \frac{1}{x^2} = 1 \quad \Rightarrow \quad \frac{2}{x^2} = 1 \quad \Rightarrow \quad x = \pm\sqrt{2}.$$

Then, $y = \mp\sqrt{2}$, that is, the points to consider are

$$(\sqrt{2}, -\sqrt{2}), \qquad (-\sqrt{2}, \sqrt{2}).$$

$$f(\sqrt{2}, -\sqrt{2}) = -\frac{1}{\sqrt{2}} + \frac{1}{(-\sqrt{2})} = -\frac{2}{\sqrt{2}} = -\sqrt{2},$$
$$f(-\sqrt{2}, \sqrt{2}) = -\frac{1}{(-\sqrt{2})} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Therefore, we conclude that

$$\boxed{(\sqrt{2}, -\sqrt{2}) \text{ is a minimum of } f},$$
$$\boxed{(-\sqrt{2}, \sqrt{2}) \text{ is a maximum of } f}.$$

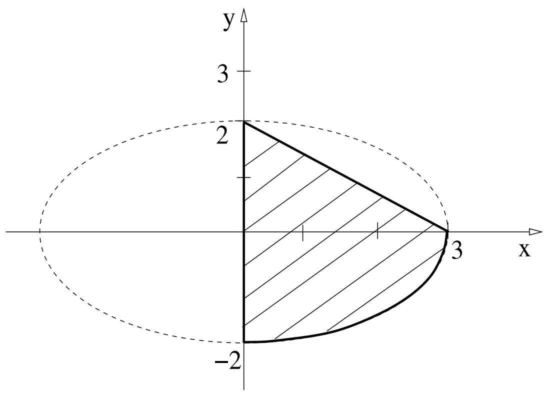
7. Consider the integral $\displaystyle\int\int_D f(x,y)\,dA = \int_0^3 \int_{-2\sqrt{1-\frac{x^2}{3^2}}}^{2(1-\frac{x}{3})} f(x,y)\,dy\,dx.$

(a) (8 points) Sketch the region of integration.

(b) (8 points) Switch the order of integration in the above integral.

(c) (8 points) Compute the integral $\displaystyle\int\int_D f(x,y)\,dA$ for the case $f(x,y) = xy$.

(a)



(b)

$$\int\int_D f(x,y)\,dA = \int_{-2}^{0} \int_0^{3\sqrt{1-\frac{y^2}{2^2}}} f(x,y)\,dx\,dy + \int_0^2 \int_0^{3(1-\frac{y}{2})} f(x,y)\,dx\,dy.$$

(c)

$$I = \int_0^3 \int_{-2\sqrt{1-\frac{x^2}{3^2}}}^{2(1-\frac{x}{3})} xy\,dy\,dx = \int_0^3 x\left(\frac{y^2}{2}\Big|_{-2\sqrt{1-\frac{x^2}{3^2}}}^{2(1-\frac{x}{3})}\right)dx,$$

$$I = \frac{1}{2}\int_0^3 x\left[4\left(1-\frac{x}{3}\right)^2 - 4\left(1-\frac{x^2}{3^2}\right)\right]dx = 2\int_0^3 x\left(1+\frac{x^2}{3^2} - 2\frac{x}{3} - 1 + \frac{x^2}{3^2}\right)dx,$$

$$I = 2\int_0^3 x\left(2\frac{x^2}{3^2} - 2\frac{x}{3}\right)dx = \frac{4}{3^2}\int_0^3 \left(x^3 - 3x^2\right)dx,$$

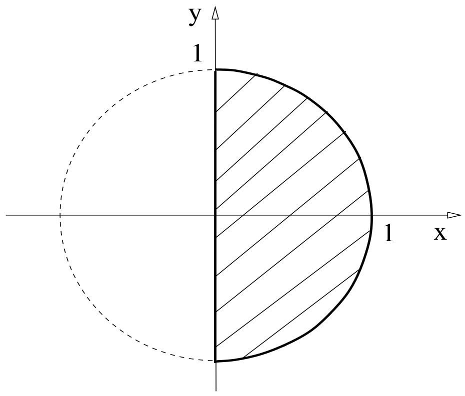
$$I = \frac{4}{3^2}\left(\frac{x^4}{4} - x^3\right)\Big|_0^3 = \frac{4}{3^2}\left(\frac{3^4}{4} - 3^3\right) = 4\left(\frac{3^2}{4} - 3\right),$$

$$I = (9-12) = -5, \quad \Rightarrow \quad \boxed{I = -5}.$$

8. (10 points) Transform to polar coordinates and then evaluate the integral

$$I = \int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \left(x^2 + y^2\right)^{3/2} dx\, dy.$$

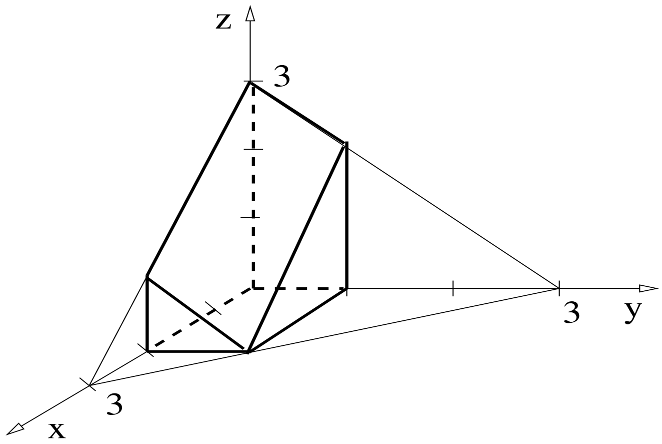
The integration region is given by



therefore, the integral I in polar coordinates is the following

$$\begin{aligned}
I &= \int_{-\pi/2}^{\pi/2} \int_{0}^{1} \left(r^2\right)^{3/2} (r\, dr)\, d\theta, \\
&= \left(\int_{-\pi/2}^{\pi/2} d\theta\right) \left(\int_{0}^{1} r^4\, dr\right), \\
&= \pi \left(\frac{r^5}{5}\Big|_0^1 \right), \\
&= \frac{\pi}{5}, \quad \Rightarrow \quad \boxed{I = \frac{\pi}{5}}.
\end{aligned}$$

9. (10 points) Find the volume of a parallelepiped whose base is a rectangle in the $z = 0$ plane given by $0 \leqslant y \leqslant 1$ and $0 \leqslant x \leqslant 2$, while the top side lies in the plane $x+y+z = 3$.



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\begin{aligned}
V &= \int_0^2 \int_0^1 \int_0^{3-x-y} dz\, dy\, dx \\
&= \int_0^2 \int_0^1 (3 - x - y)\, dy\, dx, \\
&= \int_0^2 \left[(3 - x)\left(y\Big|_0^1 \right) - \frac{1}{2}\left(y^2\Big|_0^1 \right) \right] dx, \\
&= \int_0^2 \left(3 - x - \frac{1}{2} \right) dx, \\
&= \int_0^2 \left(\frac{5}{2} - x \right) dx, \\
&= \left[\frac{5}{2}\left(x\Big|_0^2 \right) - \frac{1}{2}\left(x^2\Big|_0^2 \right) \right], \\
&= 5 - 2, \\
&= 3 \quad \Rightarrow \quad \boxed{V = 3}.
\end{aligned}
$$