Slide 1

Functions defined using infinite series

- Power series.
- The convergence of power series.
- Differentiation and integration.

Slide 2

The geometric series can be used to define a function

We have learned how to add infinitely many terms.

We can use this knowledge to define functions.

$$f(x) = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.$$

In this case we know the explicit expression for the sum:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.$$

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A power series is an infinite sum of power functions

Definition 1 *A power series centered at* $x = 0$ *is given by*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

A power series centered at $x = a$ *is given by*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where a, c_n *are constants.*

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Not every function constructed with an infinite series is a power series

Consider the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, which converges for $p > 1$.

(By integral test, although the number that it converges to is not know exactly.)

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^x, \quad x \in (1, \infty),$$

converges, but it is not a power series.

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Here is a simple example of a power series

$$f(x) = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n, \quad 0 < x < 4.$$

Show that for $0 < x < 4$ holds

$$\frac{2}{x} = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \frac{1}{8}(x-2)^3 + \cdots,$$

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What is the interval in x where $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges?

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots,$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots,$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

$$\sum_{n=0}^{\infty} n!\, x^n = 1 + x + 2!\, x^2 + 3!\, x^3 + \cdots.$$

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Summary about the convergence of power series

Theorem 1 *The convergence of* $\sum_{n-0}^{\infty} c_n(x-a)^n$ *is described by one of the following cases:*

- *Exists* $R > 0$ *such that the series converges for* $|x-a| < R$ *and diverges for* $|x-a| > R$. *The series may or may not converge at the endpoints* $x = a + R$ *and* $x = a - R$.
- *The series converges for all* $x \in I\!R$ $(R = \infty)$.
- *The series converges only for* $x = a$ $(R = 0)$.

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We introduce here the radius and interval of convergence

In the above formulas R is called the radius of convergence.

The interval of radius R centered at $x = a$ where the series converges is called interval of convergence.

This interval can be open, closed, of half open.

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Differentiation and integration of power series is done term by term

Theorem 2 *Suppose that* $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ *converges for* $|x-a| < R$*, with* $R > 0$*. Then,*

$$f'(x) = \sum_{n=0}^{\infty} n\, c_n (x-a)^{n-1},$$

$$\int f(x)\, dx = \sum_{n=0}^{\infty} \frac{c_n}{(n+1)} (x-a)^{(n+1)} + c.$$

Both $f'(x)$ *and* $\int f(x)\, dx$ *converges for* $|x-a| < R$*.*

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Taylor polynomials and Taylor series

- Review: Differentiation and integration of power series.
- Taylor polynomials and series of a function.
- Convergence of Taylor series.

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Differentiation and integration of power series is done term by term

Theorem 3 *Suppose that* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *converges for* $|x-a| < R$*, with* $R > 0$*. Then,*

$$\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right)' = \sum_{n=0}^{\infty} \left(c_n(x-a)^n\right)',$$

$$\int \left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) dx = \sum_{n=0}^{\infty} \int \left(c_n(x-a)^n\right) dx.$$

Both right hand sides converge for $|x-a| < R$*.*

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We summarize some ideas so far and formulate our next big problem

Power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ are well defined functions where they converge. Say for x such that $0 \leq |x-a| \leq R$.

Power series functions have ∞-many derivatives.

Power series functions agree with previously known functions on some intervals.

Has any function with ∞-many derivatives a power series expression, at least in some interval?

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Given a function $f(x)$, how can the coefficients of a power series be constructed?

A key idea for the answer is to see what actually happens with power series functions.

Theorem 4 *Consider a power series function $\sum_{n=0}^{\infty} c_n(x-a)^n$ that converges for $0 < |x-a| < R$. Denote*

$$\sum_{n=0}^{\infty} c_n(x-a)^n = f(x).$$

Then,

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

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The previous formula suggests what is a candidate for a power series of a given function $f(x)$

Definition 2 *Let $f(x)$ be a function having ∞-many derivatives at $x = a$. The Taylor series generated by $f(x)$ at $x = a$ is defined as*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

For what x does the Taylor series converge?

Does it converge to $f(x)$?

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Let us start defining a polynomial in n, and later on we study the limit $n \to \infty$

Definition 3 *Let $f(x)$ be a function having n derivatives at $x = a$. The Taylor polynomial of order n generated by $f(x)$ at $x = a$ is defined as*

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k.$$

The degree of $T_n(x)$ is $\leq n$, because $f^{(n)}(a)$ could be zero.

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The Taylor series of a function is well defined when the remainder $R_n(x) \to 0$ as $n \to \infty$

If $f(x)$ having n derivatives then $T_n(x)$ is well defined.

The remainder $R_n(x)$ is defined by the equation

$$f(x) = T_n(x) + R_n(x).$$

$$T_n(x) \to \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n, \qquad \text{as} \qquad n \to \infty.$$

$f(x)$ **has a series representation** $\Leftrightarrow \lim_{n\to\infty} R_n(x) = 0$

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Here is what $R_n(x)$ looks like

Theorem 5 *Let f, f', $\cdots$, $f^{(n+1)}$ be continuous in $0 < |x - a| < R$. Then*

$$f(x) = T_n(x) + R_n(x),$$

with $T_n(x)$ the Taylor polynomial of order n and

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\,(x-a)^{(n+1)}, \quad 0 < |a - c| < R.$$

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Here is what is needed on the remainder $R_n(x)$ in order it tends to zero for large n

Theorem 6 *Let f, f', $\cdots$, $f^{(n)}$ be continuous and satisfy $|f^{(n)}(x)| < M$ for all $n \geq 0$ and $0 < |x - a| < R$. Then*

$$\lim_{n\to\infty} |R_n(x)| = 0$$

for $0 < |x - a| < R$ and then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\,(x-a)^n.$$

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Taylor polynomials approximate functions

- Review: Convergence of Taylor series.
- Taylor polynomials to approximate functions.

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The Taylor series of a function is well defined when the remainder $R_n(x) \to 0$ as $n \to \infty$

If $f(x)$ having n derivatives then $T_n(x)$ is well defined.

The remainder $R_n(x)$ is defined by the equation

$$f(x) = T_n(x) + R_n(x).$$

$$T_n(x) \to \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n, \quad \text{as} \quad n \to \infty.$$

$f(x)$ **has a series representation** $\Leftrightarrow \lim_{n\to\infty} R_n(x) = 0$

Slide 21

Here is what is needed on the remainder $R_n(x)$ in order it tends to zero for large n

Theorem 7 *Let f, f', $\cdots$, $f^{(n)}$ be continuous and satisfy $|f^{(n)}(x)| < M$ for all $n \geq 0$ and $0 < |x - a| < R$. Then*

$$\lim_{n\to\infty} |R_n(x)| = 0$$

for $0 < |x - a| < R$ and then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

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A useful limit to verify whether Taylor series converges

Given any number $a \in \mathbb{R}$, the following limit holds,

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0,$$

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Taylor polynomials approximate functions with polynomials

Consider the following example:

The energy of a free particle with rest mass m and velocity v is given by Einstein's formula

$$E(v) = \frac{mc^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

where c is the speed of light.

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Einsteinian and Newtonian kinetic energies have very different expressions

The Einstein kinetic energy is the difference between $E(v)$ and the rest energy $E(0)$,

$$E_K(v) = \frac{mc^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} - mc^2.$$

The Newtonian kinetic energy is

$$N_K(v) = \frac{1}{2}mv^2.$$

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Newtonian kinetic energy is an approximation of the Einstein's kinetic energy

$$\frac{mc^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = T_2(v) + R_2(v),$$

where

$$T_2(v) = mc^2 + \frac{1}{2}mv^2,$$

Therefore we get that Newton kinetic energy is the second Taylor polynomial approximation of Einstein's kinetic energy:

$$N_K(v) = \frac{1}{2}mv^2.$$

The approximation is by a second Taylor polynomial

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Introduction to differential equations

- Examples of differential equations.
- Separable differential equations.
- Examples and applications.

Slide 27

Physics describes nature through differential equations

Newton's law of movement of a particle.
Unknown: $x(t)$, position as function of time.
Equation: (mass times acceleration equal force)

$$m\,x''(t) = F(t, x(t)).$$

where m is the mass of the particle, and F is the force applied to the particle.

Maxwell equations for electromagnetism, Schrödinger equations for quantum mechanics

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Physics describes nature through differential equations

Radioactive decay of a substance.
Unknown: $m(t)$, mass as function of time.
Equation: (The mass decay rate is proportional to the actual mass)

$$m'(t) = -k\,m(t), \quad 0 < k.$$

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Physics describes nature through differential equations

Population growth in biological systems.
Unknown: $P(t)$ number of individuals as function of time.
Equation:

$$P'(t) = k\,P(t), \quad k > 0.$$

Population growth with limited resources:

$$P'(t) = k\,P(t)\left(1 - \frac{P(t)}{K}\right), \quad k > 0,\ K > 0..$$

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Separable equations are easy to integrate

General differential equation of first order:

$$y'(x) = f(x, y(x)).$$

Definition 4 *A differential equation is separable if t has the form*

$$y'(x) = \frac{g(x)}{h(y(x))}.$$

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Separable equations are easy to integrate

Theorem 8 *Let $H(u)$ and $G(x)$ be differentiable functions, and let $H'(u) = h(u)$, and $G'(x) = g(x)$, be continuous functions. Let $H(u)$ be invertible. Then, the separable equation*

$$y'(x) = \frac{g(x)}{h(y(x))}$$

has the solution

$$y(x) = H^{-1}(G(x) + c), \quad c \text{ constant.}$$

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To find family of curves orthogonal to another family of curves is an application of differential equations

Given a family of curves $\tilde{y}(x)$, find another family of curves $y(x)$ orthogonal to $\tilde{y}(x)$, that is

$$y'(x) = -\frac{1}{\tilde{y}(x)}.$$

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Separable differential equations

- Review: Separable equations.
- Application: Find orthogonal trajectories.
- The Logistic equation.

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Separable equations are a particular class of differential equations.

Definition 5 *A differential equation is separable if t has the form*

$$y'(x) = \frac{g(x)}{h(y(x))}, \quad h(y(x)) \neq 0.$$

Examples:

$$y'(x) = \cos(x)[y(x)]^2, \quad P'(t) = kP(t)\left(1 - \frac{P(t)}{K}\right).$$

Not separable equation:

$$y'(x) = \cos(x\,y(x)).$$

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There is a general formula to integrate separable equations

Theorem 9 *Let $h(u) \neq 0$ and $g(x)$ be continuous function. Introduce their antiderivatives*

$$H(u) = \int_{u_0}^{u} h(s)\, ds, \quad G(x) = \int_{x_0}^{x} g(s)\, ds.$$

Assume that $H(u)$ is invertible. Then, the separable equation

$$y'(x) = \frac{g(x)}{h(y(x))}, \quad y(x_0) = y_0,$$

has the solution $y(x) = H^{-1}(G(x))$, with $H(y_0) = 0$, $G(x_0) = 0$.

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Application: Find curves orthogonal to a family of curves

Given a family of curves $\tilde{y}(x)$, find another family of curves $y(x)$ orthogonal to $\tilde{y}(x)$.

Theorem 10 *The curve $y(x)$ is orthogonal to the curve $\tilde{y}(x)$ at $x \Leftrightarrow$*

$$y'(x) = -\frac{1}{\tilde{y}(x)}.$$

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Orthogonal curves can be found following three main steps

Find a differential equation obeyed by $\tilde{y}(x)$.

$y(x)$ is orthogonal to the curve $\tilde{y}(x)$ at $x \Leftrightarrow$
$y'(x) = -1/y(x)$. That gives a differential equation for $y(x)$.

Solve the differential equation for $y(t)$.

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The Logistic equations has two main sets of solutions

The logistic equation describes population growth with limited resources. The equation is:

$$P'(t) = k\,P(t)\left(1 - \frac{P(t)}{K}\right), \quad P(t=0) = P_0.$$

with $k > 0$, and $K > 0$.

The two types of solutions are increasing solutions, and decreasing solutions.

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Here is the general solution to the Logistic equation

The solution is:

$$P(t) = \frac{K}{1 + Ae^{-kt}}, \quad A = \frac{K - P_0}{P_0}.$$

where $P(t = 0) = P_0$.