

The geometric series can be used to define a function

We have learned how to add infinitely many terms. We can use this knowledge to define functions.

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$$f(x) = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.$$

In this case we know the explicit expression for the sum:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.$$

A power series is an infinite sum of power functions

Definition 1 A power series centered at x = 0 is given by

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

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A power series centered at
$$x = a$$
 is given by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

where a, c_n are constants.

Not every function constructed with an infinite series is a power series

Consider the *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, which converges for p > 1.

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(By integral test, although the number that it converges to is not know exactly.)

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^x, \quad x \in (1,\infty),$$

converges, but it is not a power series.

Here is a simple example of a power series

$$f(x) = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n, \quad 0 < x < 4.$$

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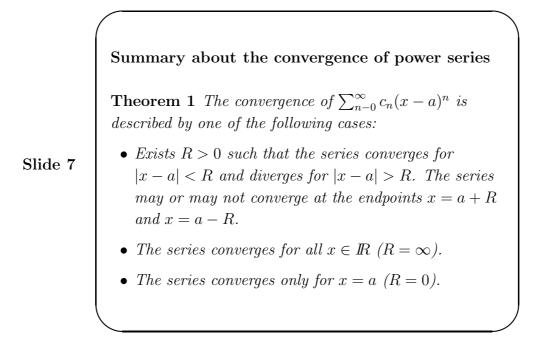
Show that for 0 < x < 4 holds

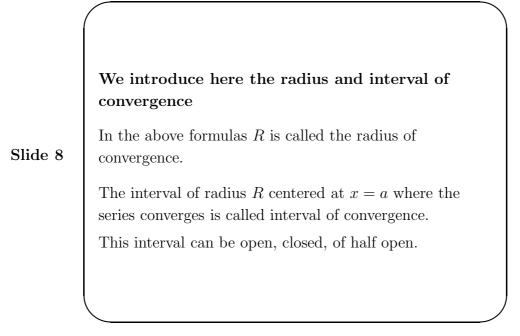
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$$\frac{2}{x} = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \frac{1}{8}(x-2)^3 + \cdots,$$

What is the interval in x where $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges?

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots,$$
$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots,$$
$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$
$$\sum_{n=0}^{\infty} n! \, x^n = 1 + x + 2! \, x^2 + 3! \, x^3 + \cdots.$$





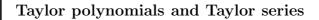
Differentiation and integration of power series is done term by term

Theorem 2 Suppose that $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges for |x-a| < R, with R > 0. Then,

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$$f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1},$$
$$\int f(x) \, dx = \sum_{n=0}^{\infty} \frac{c_n}{(n+1)} (x-a)^{(n+1)} + c.$$

Both f'(x) and $\int f(x) dx$ converges for |x - a| < R.



- Slide 10
- Review: Differentiation and integration of power series.
- Taylor polynomials and series of a function.
- Convergence of Taylor series.

Differentiation and integration of power series is done term by term

Theorem 3 Suppose that $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges for |x-a| < R, with R > 0. Then,

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$$\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right)' = \sum_{n=0}^{\infty} \left(c_n (x-a)^n\right)',$$
$$\int \left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) dx = \sum_{n=0}^{\infty} \int \left(c_n (x-a)^n\right) dx.$$
Both right hand sides converge for $|x-a| < R.$

We summarize some ideas so far and formulate our next big problem

Power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ are well defined functions where they converge. Say for x such that $0 \le |x-a| \le R$.

Slide 12 Power series functions have ∞ -many derivatives.

Power series functions agree with previously known functions on some intervals.

Has any function with ∞ -many derivatives a power series expression, at least in some interval?

Given a function f(x), how can the coefficients of a power series be constructed?

A key idea for the answer is to see what actually happens with power series functions.

Theorem 4 Consider a power series function $\sum_{n=0}^{\infty} c_n (x-a)^n$ that converges for 0 < |x-a| < R. Denote

$$\sum_{n=0}^{\infty} c_n (x-a)^n = f(x).$$

Then,

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

The previous formula suggests what is a candidate for a power series of a given function f(x)

Definition 2 Let f(x) be a function having ∞ -many derivatives at x = a. The Taylor series generated by f(x) at x = a is defined as

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$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \, (x-a)^n$$

For what x does the Taylor series converge?

Does it converge to f(x)?

Let us start defining a polynomial in n, and later on we study the limit $n \to \infty$

Definition 3 Let f(x) be a function having n derivatives at x = a. The Taylor polynomial of order n generated by f(x) at x = a is defined as

$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$$

The degree of $T_n(x)$ is $\leq n$, because $f^{(n)}(a)$ could be zero.

The Taylor series of a function is well defined when the remainder $R_n(x) \to 0$ as $n \to \infty$

If f(x) having *n* derivatives then $T_n(x)$ is well defined. The remainder $R_n(x)$ is defined by the equation

$$f(x) = T_n(x) + R_n(x)$$

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$$T_n(x) \to \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n, \quad \text{as} \quad n \to \infty.$$

f(x) has a series representation $\Leftrightarrow \lim_{n\to\infty} R_n(x) = 0$

Here is what $R_n(x)$ looks like

Theorem 5 Let $f, f', \dots, f^{(n+1)}$ be continuous in 0 < |x-a| < R. Then

$$f(x) = T_n(x) + R_n(x),$$

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with $T_n(x)$ the Taylor polynomial of order n and

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{(n+1)}, \quad 0 < |a-c| < R.$$

Here is what is needed on the remainder $R_n(x)$ in order it tends to zero for large n

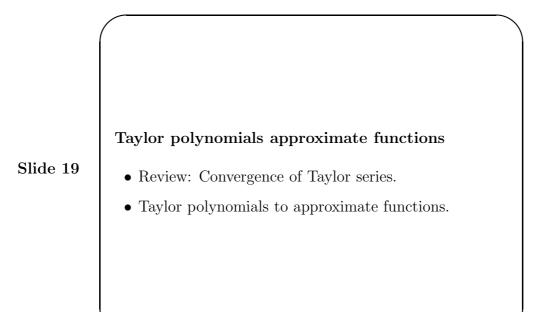
Theorem 6 Let $f, f', \dots, f^{(n)}$ be continuous and satisfy $|f^{(n)}(x)| < M$ for all $n \ge 0$ and 0 < |x - a| < R. Then

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$$\lim_{n \to \infty} |R_n(x)| = 0$$

for 0 < |x - a| < R and then $f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}}{n!}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$



The Taylor series of a function is well defined when the remainder $R_n(x) \to 0$ as $n \to \infty$

If f(x) having *n* derivatives then $T_n(x)$ is well defined. The remainder $R_n(x)$ is defined by the equation

$$f(x) = T_n(x) + R_n(x)$$

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$$T_n(x) \to \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n, \quad \text{as} \quad n \to \infty.$$

f(x) has a series representation $\Leftrightarrow \lim_{n\to\infty} R_n(x) = 0$

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Here is what is needed on the remainder $R_n(x)$ in order it tends to zero for large n

Theorem 7 Let $f, f', \dots, f^{(n)}$ be continuous and satisfy $|f^{(n)}(x)| < M$ for all $n \ge 0$ and 0 < |x - a| < R. Then

$$\lim_{n \to \infty} |R_n(x)| = 0$$

for 0 < |x - a| < R and then $\int_{-\infty}^{\infty} f^{(n)}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

A useful limit to verify whether Taylor series converges

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Given any number $a \in \mathbb{R}$, the following limit holds,

$$\lim_{n \to \infty} \frac{a^n}{n!} = 0,$$

Taylor polynomials approximate functions with polynomials

Consider the following example:

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The energy of a free particle with rest mass m and velocity v is given by Einstein's formula

$$E(v) = \frac{mc^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

where c is the speed of light.

Einsteinian and Newtonian kinetic energies have very different expressions

The Einstein kinetic energy is the difference between E(v) and the rest energy E(0),

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$$E_K(v) = \frac{mc^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} - mc^2.$$

The Newtonian kinetic energy is

$$N_K(v) = \frac{1}{2}mv^2.$$

Newtonian kinetic energy is an approximation of the Einstein's kinetic energy

$$\frac{mc^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = T_2(v) + R_2(v),$$

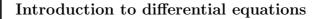
where

$$T_2(v) = mc^2 + \frac{1}{2}mv^2,$$

Therefore we get that Newton kinetic energy is the second Taylor polynomial approximation of Einstein's kinetic energy:

$$N_K(v) = \frac{1}{2}mv^2.$$

The approximation is by a second Taylor polynomial



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• Separable differential equations.

• Examples of differential equations.

• Examples and applications.

Physics describes nature through differential equations

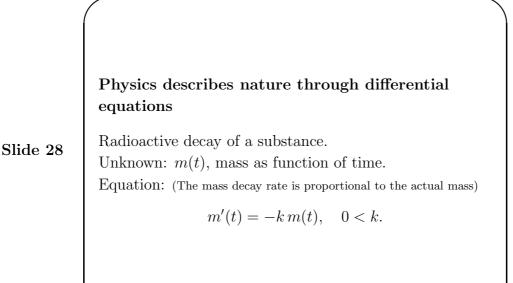
Newton's law of movement of a particle. Unknown: x(t), position as function of time. Equation: (mass times acceleration equal force)

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where m is the mass of the particle, and F is the force applied to the particle.

m x''(t) = F(t, x(t)).

Maxwell equations for electromagnetism, Schrödinger equations for quantum mechanics



Physics describes nature through differential equations

Population growth in biological systems. Unknown: P(t) number of individuals as function of time. Equation:

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$$P'(t) = k P(t), \quad k > 0.$$

Population growth with limited resources:

$$P'(t) = k P(t) \left(1 - \frac{P(t)}{K}\right), \quad k > 0, \ K > 0..$$

Separable equations are easy to integrate

General differential equation of first order:

$$y'(x) = f(x, y(x))$$

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Definition 4 A differential equation is separable if t has the form

$$y'(x) = \frac{g(x)}{h(y(x))}.$$

Separable equations are easy to integrate

Theorem 8 Let H(u) and G(x) be differentiable functions, and let H'(u) = h(u), and G'(x) = g(x), be continuous functions. Let H(u) be invertible. Then, the separable equation

$$y'(x) = \frac{g(x)}{h(y(x))}$$

has the solution

$$y(x) = H^{-1}(G(x) + c), \quad c \text{ constant}$$

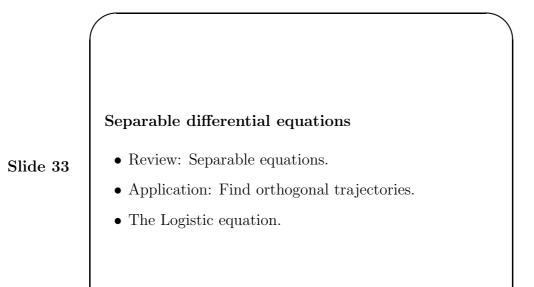
To find family of curves orthogonal to another family of curves is an application of differential equations

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Given a family of curves $\tilde{y}(x)$, find another family of curves y(x) orthogonal to $\tilde{y}(x)$, that is

$$y'(x) = -\frac{1}{\tilde{y}(x)}$$



Separable equations are a particular class of differential equations.

Definition 5 A differential equation is separable if t has the form

$$y'(x) = \frac{g(x)}{h(y(x))}, \quad h(y(x)) \neq 0.$$

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Examples:

$$y'(x) = \cos(x)[y(x)]^2$$
, $P'(t) = kP(t)\left(1 - \frac{P(t)}{K}\right)$

Not separable equation:

$$y'(x) = \cos(x \, y(x)).$$

There is a general formula to integrate separable equations

Theorem 9 Let $h(u) \neq 0$ and g(x) be continuous function. Introduce their antiderivatives

$$H(u) = \int_{u_0}^{u} h(s) \, ds, \quad G(x) = \int_{x_0}^{x} g(s) \, ds.$$

Assume that H(u) is invertible. Then, the separable equation

$$y'(x) = \frac{g(x)}{h(y(x))}, \quad y(x_0) = y_0,$$

has the solution $y(x) = H^{-1}(G(x))$, with $H(y_0) = 0$, $G(x_0) = 0$.

Application: Find curves orthogonal to a family of curves

Given a family of curves $\tilde{y}(x)$, find another family of curves y(x) orthogonal to $\tilde{y}(x)$.

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Theorem 10 The curve y(x) is orthogonal to the curve $\tilde{y}(x)$ at $x \Leftrightarrow$

$$y'(x) = -\frac{1}{\tilde{y}(x)}$$

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Orthogonal curves can be found following three main steps Find a differential equation obeyed by $\tilde{y}(x)$. y(x) is orthogonal to the curve $\tilde{y}(x)$ at $x \Leftrightarrow$ y'(x) = -1/y(x). That gives a differential equation for y(x). Solve the differential equation for y(t).

The Logistic equations has two main sets of solutions

The logistic equation describes population growth with limited resources. The equation is:

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$$P'(t) = k P(t) \left(1 - \frac{P(t)}{K} \right), \quad P(t=0) = P_0.$$

with k > 0, and K > 0.

The two types of solutions are increasing solutions, and decreasing solutions.

Here is the general solution to the Logistic equation

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The solution is:

$$P(t) = \frac{K}{1 + Ae^{-kt}}, \quad A = \frac{K - P_0}{P_0}$$

 $P(t) = \frac{1}{1+1}$ where $P(t=0) = P_0$.