Slide 1

Criterias for convergence of series

- Review: A rough criteria.
- Integral test for convergence.
- Estimating the remainder.
- Comparison test for convergence.

Slide 2

Here is a very rough criteria to check if a series diverges

Theorem 1 *If* $\lim_{n\to\infty} a_n$ *does not exist or if* $\lim_{n\to\infty} a_n \neq 0$, *then* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

It is based in the following result:

Theorem 2 *If* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Slide 3

Here is a test of convergence for certain series using integration

Theorem 3 *Let $f(x)$ be a continuous, positive, decreasing function on $[1, \infty)$. Consider the sequence $a_n = f(n)$, $n \in \mathbb{Z}^+$. Then,*
$\sum_{n=1}^{\infty} a_n$ is convergent $\Leftrightarrow \int_1^{\infty} f(x)\,dx$ is convergent.

The proof is to show the following inequality

$$\int_1^{n+1} f(x)\,dx \leq s_n \leq a_1 + \int_1^{n} f(x)\,dx.$$

Slide 4

We already used this test in the harmonic series

The harmonic series $s_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges, because

$$\ln(n+1) = \int_1^{n+1} \frac{1}{x}\,dx \leq s_n,$$

and $\ln(n+1)$ diverges when $n \to \infty$.

Slide 5

This test can be applied to the geometric series

Consider the geometric series
$s_n = \sum_{k=1}^{n} r^{(k-1)} = 1 + r + r^2 + r^3 + \cdots + r^{n-1}$,
for $0 < r < 1$.

Here $f(x) = r^x$. Recall $\int r^x \, dx = \frac{r^x}{\ln(r)} + c$. Then,

$$s_n \leq 1 + \int_1^n r^x \, dx = 1 + \frac{r}{\ln(1/r)} - \frac{r^n}{\ln(1/r)}.$$

and $r^n \to 0$ when $n \to \infty$. So, s_n converges.

We knew that the geometric series converges, $\sum_{k=1}^{\infty} r^{(k-1)} = \frac{1}{1-r}$.

Slide 6

This test can be applied to the p-series

Theorem 4 *The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$, and it diverges for $p \leq 1$.*

$$\int_1^{\infty} \frac{1}{x^p} \, dx = \begin{cases} \frac{1}{p-1} & p > 1 \\ \text{diverges} & p \leq 1 \end{cases}$$

Slide 7

The remainder of a series can be estimated with the same type of integral bounds as above

Definition 1 *If* $s_n = \sum_{k=1}^{n} a_k$ *converges to* s*, then the remainder after* n *terms is*

$$R_n = s - s_n.$$

So, $R_n = a_{n+1} + a_{n+2} + \cdots$, that is,

$$R_n = \sum_{k=n+1}^{\infty} a_k.$$

Slide 8

Here is the estimation for the remainder

Theorem 5 *Let* $f(x)$ *be continuous, positive, decreasing function on* $[n, \infty)$*. Consider the sequence* $a_k = f(k)$ *for* $k \geq n$*. Suppose that* $\sum_{k=1}^{\infty} a_k$ *is convergent. Then,*

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$

Therefore,

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\,dx.$$

Slide 9

There are three types of comparison tests

Theorem 6 *Let* $\sum_{n=1}^{\infty} a_n$*, and* $\sum_{n=1}^{\infty} b_n$ *be series with positive terms.*

1. *If* $a_n \leq b_n$ *and* $\sum_{n=1}^{\infty} b_n$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*
2. *If* $b_n \leq a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *diverges, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*
3. *If* $\lim_{n\to\infty} a_n/b_n = c > 0$*, finite, then both* $\sum_{n=1}^{\infty} b_n$ *and* $\sum_{n=1}^{\infty} a_n$ *converge, or both diverge.*

Slide 10

More and more and more convergence tests

- Review: Integral and test.
- Alternating series.
- Absolute convergence
- Root and ratio tests.

Why so many tests? Sometimes works one, sometimes another.

Slide 11

The integral test links series convergence with and improper integral convergence

$f(x)$ continuous, positive, and decreasing function on $[1, \infty)$. Define the sequence $a_n = f(n)$, $n \in \mathbb{Z}^+$.

$\sum_{n=1}^{\infty} a_n$ is convergent $\Leftrightarrow \int_1^{\infty} f(x)\,dx$ is convergent.

The proof is to show the following inequality

$$\int_1^{n+1} f(x)\,dx \leq s_n \leq a_1 + \int_1^n f(x)\,dx.$$

Slide 12

The integral test does not apply on every series

Definition 2 *Given* $0 < a_n$*, the series* $s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$ *is called alternating series.*

That is

$$s_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1} a_n.$$

Here is the alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Slide 13

The harmonic series diverges, however the alternating harmonic series converges

Theorem 7 *If the alternating series* $\sum_{k=1}^{n}(-1)^{k+1}a_k$ *with* $a_k > 0$ *also satisfies*

- $a_{n+1} < a_n$ (decreasing sequence),
- $a_n \to 0$ *as* $n \to \infty$,

then $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ *converges.*

Slide 14

The convergence of a nonpositive series could be determined by its absolute value series

Definition 3 *A series* $\sum a_n$ *is absolutely convergent if* $\sum |a_n|$ *is convergent.*

Is the series below convergent?

$$s_n = \sum_{k=1}^{n} \frac{\cos(3k)}{k^2}.$$

First notice that it is absolutely convergent.

Slide 15

Absolutely convergence implies convergence

Theorem 8 *If* $\sum_{n=1}^{\infty} |a_n|$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Therefore, the series

$$s_n = \sum_{k=1}^{n} \frac{\cos(3k)}{k^2}.$$

is indeed convergent.

Slide 16

The root test for a series is to compare it with a geometric series

Theorem 9 *Consider the series* $\sum_{k=1}^{n} a_k$ *and denote*

$$L = \lim_{n\to\infty} |a_n|^{1/n}.$$

- *If* $L < 1$*, then* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*
- *If* $L > 1$ *or* L *is* ∞*, then* $\sum_{n=1}^{\infty} a_n$ *is divergent.*
- *If* $L = 1$*, then the test is inconclusive.*

Slide 17

The ratio test is perhaps the test one should try first

Theorem 10 *Consider the series* $\sum_{k=1}^{n} a_k$ *and denote*

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}.$$

- *If* $L < 1$*, then* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*
- *If* $L > 1$ *or* L *is* ∞*, then* $\sum_{n=1}^{\infty} a_n$ *is divergent.*
- *If* $L = 1$*, then the test is inconclusive.*