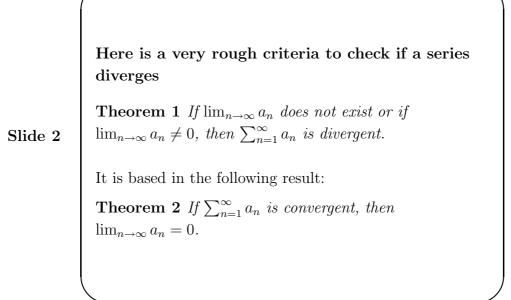


- Slide 1
- Review: A rough criteria.
- Integral test for convergence.
- Estimating the remainder.
- Comparison test for convergence.



Here is a test of convergence for certain series using integration

Slide 3

Theorem 3 Let f(x) be a continuous, positive, decreasing function on $[1, \infty)$. Consider the sequence $a_n = f(n), n \in \mathbb{Z}^+$. Then, $\sum_{n=1}^{\infty} a_n$ is convergent $\Leftrightarrow \int_1^{\infty} f(x) dx$ is convergent.

The proof is to show the following inequality

$$\int_{1}^{n+1} f(x) \, dx \le s_n \le a_1 + \int_{1}^{n} f(x) \, dx.$$

We already used this test in the harmonic series

The harmonic series $s_n = \sum_{k=1}^n \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges, because

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$$\ln(n+1) = \int_{1}^{n+1} \frac{1}{x} \, dx \le s_n$$

and $\ln(n+1)$ diverges when $n \to \infty$.

 $\begin{cases} \text{This test can be applied to the geometric series} \\ \text{Consider the geometric series} \\ s_n = \sum_{k=1}^n r^{(k-1)} = 1 + r + r^2 + r^3 + \dots + r^{n-1}, \\ \text{for } 0 < r < 1. \end{cases} \\ \text{Slide 5} \qquad \text{Here } f(x) = r^x. \text{ Recall } \int r^x \, dx = \frac{r^x}{\ln(r)} + c. \text{ Then}, \\ s_n \leq 1 + \int_1^n r^x \, dx = 1 + \frac{r}{\ln(1/r)} - \frac{r^n}{\ln(1/r)}. \\ \text{and } r^n \to 0 \text{ when } n \to \infty. \text{ So, } s_n \text{ converges.} \end{cases}$ $\text{We knew that the geometric series converges, } \sum_{k=1}^\infty r^{(k-1)} = \frac{1}{1-r}. \end{cases}$

This test can be applied to the *p*-series

Theorem 4 The *p*-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for p > 1, and it diverges for $p \le 1$.

$$\int_{1}^{\infty} \frac{1}{x^{p}} dx = \begin{cases} \frac{1}{p-1} & p > 1\\ \text{diverges} & p \le 1 \end{cases}$$

The remainder of a series can be estimated with the same type of integral bounds as above

Definition 1 If $s_n = \sum_{k=1}^n a_k$ converges to s, then the remainder after n terms is

 $R_n = s - s_n.$

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So, $R_n = a_{n+1} + a_{n+2} + \cdots$, that is,

$$R_n = \sum_{k=n+1}^{\infty} a_k$$

Here is the estimation for the remainder

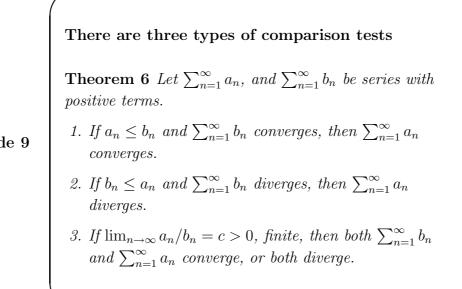
Theorem 5 Let f(x) be continuous, positive, decreasing function on $[n, \infty)$. Consider the sequence $a_k = f(k)$ for $k \ge n$. Suppose that $\sum_{k=1}^{\infty} a_k$ is convergent. Then,

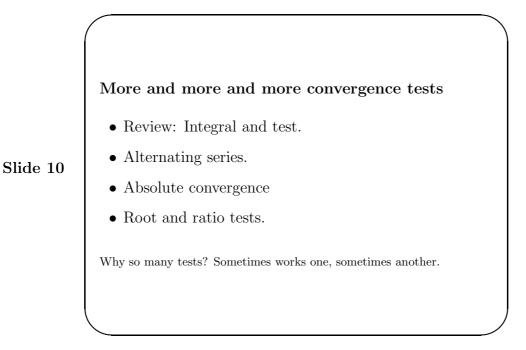
Slide 8

$$\int_{n+1}^{\infty} f(x) \, dx \le R_n \le \int_n^{\infty} f(x) \, dx.$$

Therefore,

$$s_n + \int_{n+1}^{\infty} f(x) \, dx \le s \le s_n + \int_n^{\infty} f(x) \, dx.$$





The integral test links series convergence with and improper integral convergence

f(x) continuous, positive, and decreasing function on $[1, \infty)$. Define the sequence $a_n = f(n), n \in \mathbb{Z}^+$.

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 $\sum_{n=1}^{\infty} a_n$ is convergent $\Leftrightarrow \int_1^{\infty} f(x) dx$ is convergent.

The proof is to show the following inequality

$$\int_{1}^{n+1} f(x) \, dx \le s_n \le a_1 + \int_{1}^{n} f(x) \, dx$$

The integral test does not apply on every series

Definition 2 Given $0 < a_n$, the series $s_n = \sum_{k=1}^n (-1)^{k+1} a_k$ is called alternating series.

That is

$$s_n = a_1 - a_2 + a_3 - a_4 + \dots + (-1)^{n+1} a_n$$

Here is the alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

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The harmonic series diverges, however the alternating harmonic series converges

Theorem 7 If the alternating series $\sum_{k=1}^{n} (-1)^{k+1} a_k$ with $a_k > 0$ also satisfies

- $a_{n+1} < a_n$ (decreasing sequence),

• $a_n \to 0 \text{ as } n \to \infty$, then $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges.

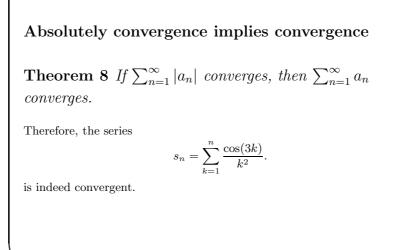
The convergence of a nonpositive series could be determined by its absolute value series

Definition 3 A series $\sum a_n$ is absolutely convergent if $\sum |a_n|$ is convergent.

Is the series below convergent?

$$s_n = \sum_{k=1}^n \frac{\cos(3k)}{k^2}.$$

First notice that it is absolutely convergent.



The root test for a series is to compare it with a geometric series **Theorem 9** Consider the series $\sum_{k=1}^{n} a_k$ and denote $L = \lim_{n \to \infty} |a_n|^{1/n}.$ If L < 1, then ∑_{n=1}[∞] a_n is absolutely convergent.
If L > 1 or L is ∞, then ∑_{n=1}[∞] a_n is divergent. • If L = 1, then the test is inconclusive.

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The ratio test is perhaps the test one should try first

Theorem 10 Consider the series $\sum_{k=1}^{n} a_k$ and denote

$$L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}$$

- If L < 1, then ∑_{n=1}[∞] a_n is absolutely convergent.
 If L > 1 or L is ∞, then ∑_{n=1}[∞] a_n is divergent.
- If L = 1, then the test is inconclusive.