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Integrals of functions on infinite domains

- Review: Improper integrals type I.
- Type II: Three main possibilities.
- Limit of an infinite sequence.

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Generalizations of $\int_a^b f(x)\,dx$ in $I = [a, b]$

Integrals on infinite domains are called improper integrals of type I

- Type I: The interval is infinite: $I = (-\infty, b]$, or $I = [a, \infty)$ or $I = (-\infty, \infty)$.

Integrals of divergent functions on finite domains are called improper integrals of type II.

- Type II: $f(x)$ is not bounded at one or more points in $[a, b]$. ($f(x)$ can have a vertical asymptote in $[a, b]$.)

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Type II: Vertical asymptote at b

Possibility (a):

Definition 1 *If $f(x)$ is continuous in $[a, b)$ then*

$$\int_a^{b^-} f(t)\,dt = \lim_{x \to b^-} \int_a^x f(t)\,dt.$$

The integral is said to converge if the limit exists and it is finite.

Otherwise the integral is said to diverge.

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Type II: Vertical asymptote at a

Possibility (b):

Definition 2 *If $f(x)$ is continuous in $(a, b]$ then*

$$\int_{a^+}^{b} f(t)\,dt = \lim_{x \to a^+} \int_x^b f(t)\,dt.$$

The integral is said to converge if the limit exists and it is finite.

Otherwise the integral is said to diverge.

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Type II: Vertical asymptote in the interior

Possibility (c):

Definition 3 *If $f(x)$ has a vertical asymptote at $c \in (a, b)$, then*

$$\int_a^b f(t)\,dt = \int_a^{c^-} f(t)\,dt + \int_{c^+}^b f(t)\,dt$$

provided that both integrals in the right hand side are convergent.

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Comparison theorem: Type I (a) case

Theorem 1 *Let $f(x)$, $g(x)$ be continuous functions for $x \geq a$ and such that $0 \leq g(x) \leq f(x)$. Then:*

- *If $\int_a^\infty f(x)\,dx$ converges $\Rightarrow$ $\int_a^\infty g(x)\,dx$ converges.*
- *If $\int_a^\infty g(x)\,dx$ diverges $\Rightarrow$ $\int_a^\infty f(x)\,dx$ diverges.*

There are analogous versions for all other cases.

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A sequence is a function whose domain are the positive integers

Definition 4 *If for every positive integer n there is associated a real number a_n, then the ordered set*

$$a_1, a_2, \cdots, a_n, \cdots$$

is called an infinite sequence. Is is denoted as $\{a_n\}$.

Definition 5 *A function $f : \mathbb{Z}^+ \to \mathbb{R}$ is called an infinite sequence.*

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The limits of sequences is the same as in functions of real numbers

Definition 6 *The sequence $\{a_n\}$ is said to have the limit L is for all $\epsilon > 0$ there exists a number $N > 0$ such that*

$$|a_n - L| < \epsilon, \quad \textit{for all } n \geq N.$$

In this case we say $\lim_{n\to\infty} a_n = n$ or $a_n \to L$ as $n \to \infty$. We say that the sequence converges.

Otherwise, we say that the sequence diverges.

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Increasing-decreasing and bounded above-below are important classes of sequences

- A sequence $\{a_n\}$ is said to be increasing $\Leftrightarrow a_n < a_{n+1}$ for all $n \geq 1$.

  A sequence $\{a_n\}$ is said to be decreasing $\Leftrightarrow a_{n+1} < a_n$ for all $n \geq 1$.

- A sequence $\{a_n\}$ is said to be bounded above $\Leftrightarrow$ exists $M > 0$ such that $a_n < M$ for all $n \geq 1$.

  A sequence $\{a_n\}$ is said to be bounded below $\Leftrightarrow$ exists $m > 0$ such that $m < a_n$ for all $n \geq 1$.

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Important tool to show that a sequence converges

- If $\{a_n\}$ is increasing and bounded above then converges.
- If $\{a_n\}$ is decreasing and bounded below then converges.

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Sequences and series

- Definitions and limits.
- Examples: Harmonic and geometric series.
- Criteria for the convergence of series.

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A sequence is a function whose domain are the positive integers

Definition 7 *A function* $f : \mathbb{Z}^{+} \to \mathbb{R}$ *is called an infinite sequence.*

Notation: $f(n)$, or f_n, or a_n, or $\{a_n\}$.

Then, for every positive integer n there is associated a real number a_n, and so the ordered set

$$a_1, a_2, \cdots, a_n, \cdots$$

is called an infinite sequence.

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Here are some of the most popular sequences

- Harmonic sequence:
$$\left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}.$$
- Geometric sequence:
$$\left\{\frac{1}{2^{(n-1)}}\right\} = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\right\}.$$
-
$$\left\{\left(1 + \frac{a}{n}\right)^n\right\}.$$

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The limits of sequences is the same as in functions of real numbers

Definition 8 *The sequence $\{a_n\}$ is said to have the limit L is for all $\epsilon > 0$ there exists a number $N > 0$ such that*
$$|a_n - L| < \epsilon, \quad \textit{for all } n \geq N.$$

In this case we say that the sequence is convergent and $\lim_{n\to\infty} a_n = n$ or $a_n \to L$ as $n \to \infty$.

Otherwise, we say that the sequence diverges.

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Here are the limits of some of the most popular sequences

- Harmonic sequence:

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

- Geometric sequence:

$$\lim_{n\to\infty} \frac{1}{2^{(n-1)}} = 0$$

- $$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a.$$

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A series is a sequence of partial sums

Definition 9 *A series is a sequence* $\{s_n\}$ *where* s_n *has the form* $s_n = \sum_{k=1}^{n} a_k$.

Given $a_1, a_2, a_3, \cdots, a_n, \cdots$, form the partial sums

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3,$$

$$s_n = a_1 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

Then, $\{s_n\}$ or equivalently is a series.

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Here are some of the most popular series

- Harmonic series:
$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$
- Geometric series:
$$\sum_{k=1}^{n} \frac{1}{2^{(k-1)}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{(n-1)}}.$$

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Limits of series involves adding infinitely many terms

Definition 10 *The sum* $\sum_{k=1}^{\infty} a_k$ *is convergent and has the value* s *if the sequence of partial sums converges to* s*, that is,*
$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_n = s.$$

If the series $\{\sum_{k=1}^{n} a_k\}$ is not convergent, we call it divergent.

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Here are the sum of some of the most popular series

- Harmonic series:

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{4} + \cdots \quad \text{diverges.}$$

- Geometric series:

$$\sum_{k=1}^{\infty} \frac{1}{2^{(k-1)}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2.$$

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This is a very rough criteria to check whether a series converges or not

First a result:

Theorem 2 *If* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Then the criteria:

Theorem 3 *If* $\lim_{n\to\infty} a_n$ *does not exist or if* $\lim_{n\to\infty} a_n \neq 0$, *then* $\sum_{n=1}^{\infty} a_n$ *is divergent.*