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Integration of rational functions

- Review: Decomposition of a polynomial.
- Integration of rational functions: Cases I - IV.
- Examples.
- The three main integrals.

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A rational function is a quotient of polynomials

$$\frac{P_n(x)}{Q_m(x)} = S_p(x) + \frac{R_q(x)}{Q_m(x)}.$$

with $p = n - m$ and $0 \leq q < m$.

Quotients of polynomials can always be integrated

They can be reduced into one of four cases

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There are four possible cases in the decomposition of a rational function

I: The denominator is a product of distinct linear factors.

II: The denominator is a product of linear factors, some of which are repeated.

III: The denominator contains irreducible quadratic factors, none of which are repeated.

IV: The denominator contains irreducible quadratic factors, some of which are repeated.

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Case I: The denominator is a product of distinct linear factors

$$\int \frac{2x^2 + 5x - 1}{x(x-1)(x+2)}\, dx.$$

Case II: The denominator is a product of linear factors, some of which are repeated

$$\int \frac{x^2 + 2x + 3}{(x-1)(x+1)^2}\, dx$$

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Case III: The denominator contains irreducible quadratic factors, none of which are repeated

$$\int \frac{3x^2 + 2x - 2}{(x-1)(x^2+x+1)}\,dx$$

Case IV: The denominator contains irreducible quadratic factors, some of which are repeated

$$\int \frac{x^4 - x^3 + 2x^2 - x + 2}{(x-1)(x^2+2)^2}\,dx$$

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Every polynomial can be decomposed into a product of polynomials of degree one and two

Theorem 1 *Every polynomial $Q_m(x)$ with $m \geq 0$ and real coefficients can be decomposed as*

$$Q_m(x) = a(x-a_1)^{\ell_1} \cdots (x-a_r)^{\ell_r} \left(x^2 + b_1\,x + c_1\right)^{m_1} \cdots \left(x^2 + b_s\,x + c_s\right)^{m_s}.$$

with $m = \ell_1 + \cdots + \ell_r + m_1 + \cdots + m_s$.

The $a_1, \cdots, a_r$ are roots of $Q_m(x)$, that is, $Q_m(a_i) = 0$, for $i = 1, \cdots, r$.

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The problem of integrate a rational function reduces to that of calculating integrals of the form:

$$I_1 = \int \frac{dx}{(x+a)^m},$$

$$I_2 = \int \frac{x\,dx}{(x^2+bx+c)^m}, \quad I_3 = \int \frac{dx}{(x^2+bx+c)^m}.$$

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The solution for I_1 is:

$$I_1 = \int \frac{dx}{(x+a)^m} = \begin{cases} \ln(|x+a|) + c & m = -1, \\ \frac{1}{(1-m)(x+a)^{m-1}} + c & m \neq -1/ \end{cases}$$

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The integrals I_2 and I_3 can be transformed into the following:

$$\tilde{I}_2 = \int \frac{u\,du}{(u^2+\alpha^2)^m}, \quad \tilde{I}_3 = \int \frac{du}{(u^2+\alpha^2)^m}.$$

with the substitution

$$u = x + \frac{b}{2}, \quad \alpha = \frac{1}{2}\sqrt{4c - b^2}.$$

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The solution of $\tilde{I}_2$ is:

$$\tilde{I}_2 = \int \frac{u\,du}{(u^2+\alpha^2)^m} = \begin{cases} \frac{1}{2}\ln(u^2+\alpha^2) + c & m = -1, \\ \frac{1}{2(1-m)(u^2+\alpha^2)^{m-1}} + c & m \neq -1/ \end{cases}$$

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The solution of $\tilde{I}_3$ is:

$$\tilde{I}_3 = \int \frac{du}{(u^2 + \alpha^2)} = \frac{1}{\alpha} \arctan\left(\frac{u}{\alpha}\right) + c, \quad m = 1.$$

The case $m > 1$ is reduced to the case $m = 1$ by the following recursive formula

$$\int \frac{du}{(u^2+\alpha^2)^m} = \frac{1}{2(m-1)\alpha^2} \frac{u}{(u^2+\alpha^2)^m} + \frac{2m-3}{2(m-1)\alpha^2} \int \frac{du}{(u^2+\alpha^2)^{(m-1)}}.$$

So, we have integrated all possible cases!

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Numerical Methods: Compute Riemann sums

- Left and right points.
- Midpoint rule.
- Trapezoidal rule.
- Simpson's rule.

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The integral of algebraic functions is not always an algebraic function

Definition of logarithm:

$$\int_1^x \frac{1}{t}\,dt =: \ln(x).$$

Definition of elliptic function:

$$\int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} =: u(x).$$

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Numerical methods just compute finite Riemann sums for different choices of sample points

Definition 1 (Riemann sum) *Let $f(x)$ be a function defined on a interval $x \in [a,b]$. The Riemann sum of order n of $f(x)$ in $[a,b]$ is the number given by*

$$R_n = \sum_{i=0}^{n-1} f(x_i^*)\,\Delta x,$$

Given a natural number n we have introduced a partition on $[a,b]$ given by $\Delta x = (b-a)/n$. We denoted $x_i^* \in [x_i, x_{i+1}]$ a sample point.

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Some methods use different sample points $x_i^* \in [x_i, x_{i+1}]$

- Left point rule: $x_i^* = x_i$.
- Right point rule: $x_i^* = x_{i+1}$.
- Midpoint rule: $x_i^* = (x_{i+1} + x_i)/2$.

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The resulting Riemann sums are the following

- Left point rule:

$$L_n = [f(x_0) + f(x_1) + \cdots + f(x_{n-1})]\Delta x.$$

- Right point rule: $x_i^* = x_{i+1}$.

$$R_n = [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x.$$

- Midpoint rule: $x_i^* = (x_{i+1} + x_i)/2$.

$$M_n = [f(x_0^*) + f(x_2^*) + \cdots + f(x_{n-1}^*)]\Delta x.$$

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Other methods compute the area below f in different ways

- Trapezoidal rule:

$$A_i = [f(x_{i+1}) + f(x_i)]\,\frac{\Delta x}{2}.$$

- Simpson's rule:

$$A_i = [f(x_{i+1}) + 4f(x_i) + f(x_{i-1})]\,\frac{\Delta x}{3}.$$

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The resulting Riemann sums are the following

- Trapezoidal rule:

$$T_n = \left[\frac{1}{2}f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right]\Delta x.$$

- Simpson's rule: ($n = 2m$ even)

$$\begin{aligned} T_8 &= [f(x_0) + 4f(x_1) \\ &\quad +2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + 2f(x_6) \\ &\quad +4f(x_7) + f(x_8)]\,\frac{\Delta x}{3}. \end{aligned}$$

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Integrals of functions on infinite domains

- Improper integrals type I and II.
- Type I: Three main possibilities.
- Examples.

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Generalizations of $\int_a^b f(x)\,dx$ in $I = [a, b]$

Integrals on infinite domains are called improper integrals of type I

- Type I: The interval is infinite: $I = (-\infty, b]$, or $I = [a, \infty)$ or $I = (-\infty, \infty)$.

Integrals of divergent functions on finite domains are called improper integrals of type II.

- Type II: $f(x)$ is not bounded at one or more points in $[a, b]$. ($f(x)$ can have a vertical asymptote in $[a, b]$.)

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Type I: Infinite domains

Possibility (a):

Definition 2 *If* $\int_a^x f(t)\,dt = F(x)$ *exists for all* $x \geq a$*, then*

$$\int_a^\infty f(t)\,dt = \lim_{x\to\infty} F(x).$$

The integral is said to converge if $\lim_{x\to\infty} F(x)$ exists and it is finite.

Otherwise the integral is said to diverge.

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Type I: Infinite domains

Possibility (b):

Definition 3 *If* $\int_x^b f(t)\,dt = F(x)$ *exists for all* $x \leq b$*, then*

$$\int_{-\infty}^b f(t)\,dt = \lim_{x\to-\infty} F(x).$$

The integral is said to converge if $\lim_{x\to-\infty} F(x)$ exists and it is finite.

Otherwise the integral is said to diverge.

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Type I: Infinite domains

Possibility (c):

Definition 4 *If both* $\int_{-\infty}^{c} f(t)\,dt$ *and* $\int_{c}^{\infty} f(t)\,dt$ *are convergent for some* $c \in \mathbb{R}$*, then*

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{c} f(t)\,dt + \int_{c}^{\infty} f(t)\,dt.$$