

Tricks for sines and cosines

Consider integrals of the form

$$\int [\sin(x)]^m [\cos(x)]^n dx$$

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If m or n is odd, then the integral can be done by substitution, recalling:

$$\sin'(x) = \cos(x), \quad \cos'(x) = -\sin(x),$$

 $\sin^2(x) + \cos^2(x) = 1.$

Tricks for sines and cosines

Consider integrals of the form

$$\int [\sin(x)]^m [\cos(x)]^n dx$$

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If both m and n are even, then integral above can be done recalling

$$\sin^2(x) = \frac{1}{2}[1 - \cos(2x)],$$
$$\cos^2(x) = \frac{1}{2}[1 + \cos(2x)].$$

Tricks for tangents and secants

Consider integrals of the form

$$\int [\tan(x)]^n [\sec(x)]^m dx.$$

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If n is odd or if m is even, then *substitution* recalling

$$\tan'(x) = \sec^2(x), \quad \sec'(x) = \tan(x) \sec(x), \\
\sec^2(x) - \tan^2(x) = 1.$$

More tricks on sines and cosines

The following integrals can be done with the associated formula:

$$\int \sin(mx)\cos(nx) \, dx \quad \to \quad \sin(\alpha)\cos(\beta) = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)];$$
$$\int \sin(mx)\sin(nx) \, dx \quad \to \quad \sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)];$$
$$\int \cos(mx)\cos(nx) \, dx \quad \to \quad \cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)].$$

 $\begin{cases} \text{Integration can be generalized to functions}}\\ f: \mathbb{R} \to \mathbb{C} \\ \text{These functions have the form } f(x) = f_1(x) + i f_2(x), \\ \text{with } f_1(x) \text{ and } f_2(x) \text{ real functions.} \\ \text{Examples:} \\ \\ f(x) = 2x + i \ln(x), \\ f'(x) = 2 + i \frac{1}{x}, \\ g(x) = \tan(3x) + i e^{2x} \\ g'(x) = 3 \sec^2(3x) + 2i e^{2x}. \end{cases}$

Riemann sums, integration, and the FTC can be generalized to function $f : \mathbb{R} \to \mathbb{C}$

$$\int f(x)dx = \int f_1(x)dx + i \int f_2(x)dx.$$

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Example:

$$\int e^{iax} dx = \frac{1}{ia} e^{iax} = -\frac{i}{a} e^{iax}.$$
$$\int [x^2 + i\cos(x)] dx = \frac{1}{3} x^3 + i\sin(x).$$

Complex tricks for sine and cosine

Euler formulas

$$e^{ix} = \cos(x) + i\sin(x),$$

$$e^{-ix} = \cos(x) - i\sin(x),$$

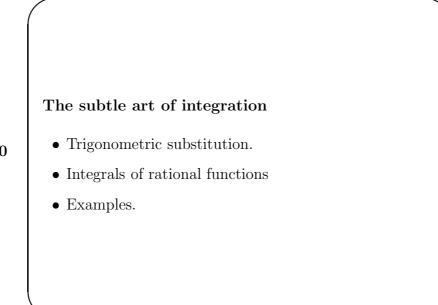
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imply that

$$\cos(x) = \frac{1}{2}[e^{ix} + e^{-ix}],$$

$$\sin(x) = \frac{1}{2i}[e^{ix} - e^{-ix}].$$

Therefore, integrals in sines and cosines can be transformed into complex integrals for exponentials.



Here is one more trick with trigs

Integrals of the form

$$\int \sqrt{a^2 - x^2} \, dx, \quad |x| \le |a|,$$

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can be computed with the change of variable:

 $x = a\sin(\theta).$

Then,

$$\int \sqrt{a^2 - x^2} \, dx = a^2 \int \cos^2(\theta) \, d\theta$$

Here are two more tricks that do not always work well, but they are worth to try

Slide 12 Integrals containing factors $\sqrt{a^2 + x^2}$ could be transformed with a change of variables $x = a \tan(\theta)$.

Integrals containing factors $\sqrt{x^2 - a^2}$, with $|x| \ge |a|$, could be transformed with a change of variables $x = a \sec(\theta)$. Quotients of polynomials can always be integrated

Definition 1 A rational function is a function of the form

$$f(x) = \frac{P_n(x)}{Q_m(x)},$$

where $P_n(x)$ and $Q_m(x)$ are a polynomials of degree nand m, respectively.

The idea is to learn how to integrate rational functions as the following ones:

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$$\int \frac{2x^3 - x^2 + x - 1}{x + 1} \, dx,$$
$$\int \frac{4x^3 - 2x^2 - x - 1}{2x^2 + x - 1} \, dx.$$

The idea is to rewrite the rational function in a more convenient way

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Theorem 1 Consider a rational function $P_n(x)/Q_m(x)$, with n > m. Then, there exist polynomials $S_p(x)$ and $R_q(x)$ of degree p = n - m and q < m such that

$$\frac{P_n(x)}{Q_m(x)} = S_p(x) + \frac{R_q(x)}{Q_m(x)}$$

The idea is to rewrite the rational function in a more convenient way

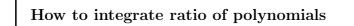
Theorem 2 Consider a rational function $R_q(x)/Q_m(x)$ with q < m, and where

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$$Q_m(x) = Q_0(x - a_1) \cdots (x - a_m),$$

with a_1, \dots, a_m are real numbers. Then, there exists real numbers A_1, \dots, A_m such that

$$\frac{R_q(x)}{Q_m(x)} = \frac{A_1}{(x - a_1)} + \dots + \frac{A_m}{(x - a_m)}$$



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- Review: Rational functions.
- Integration of rational functions.
- Examples.

A rational function is a quotient of polynomials

Definition 2 A rational function is a function of the form

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$$f(x) = \frac{P_n(x)}{Q_m(x)},$$

where $P_n(x)$ and $Q_m(x)$ are a polynomials of degree nand m, respectively.

Quotients of polynomials can always be integrated

The idea is to rewrite the rational function in a more convenient way

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Theorem 3 Consider a rational function $P_n(x)/Q_m(x)$, with n > m. Then, there exist polynomials $S_p(x)$ and $R_q(x)$ of degree p = n - m and q < m such that

$$\frac{P_n(x)}{Q_m(x)} = S_p(x) + \frac{R_q(x)}{Q_m(x)}$$

The problem of integrating a quotient of arbitrary polynomials has been simplified

One needs to study only polynomials of the form

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$$\frac{P_n(x)}{Q_m(x)}, \quad 0 \le n < m$$

This type of rational functions can be integrated using the following two results. Every polynomial can be decomposed into a product of polynomials of degree one and two Theorem 4 Every polynomial $Q_m(x)$ with $m \ge 0$ and real coefficients can be decomposed as $Q_m(x) = a(x-a_1)^{\ell_1} \cdots (x-a_r)^{\ell_r} (x^2 + b_1 x + c_1)^{m_1} \cdots (x^2 + b_s x + c_s)^{m_s}$. with $m = \ell_1 + \cdots + \ell_r + m_1 + \cdots + m_s$. The a_1, \cdots, a_r are roots of $Q_m(x)$, that is, $Q_m(a_i) = 0$, for $i = 1, \cdots, r$.

Products can be transformed into sums Theorem 5 Consider a rational function $R_q(x)/Q_m(x)$ with $0 \le q < m$, and where $Q_m(x) = a(x-a_1)^{\ell_1} \cdots (x-a_r)^{\ell_r} (x^2 + b_1 x + c_1)^{m_1} \cdots (x^2 + b_s x + c_s)^{m_s}$. Then, $R_q(x)/Q_m(x)$ can be written as $\frac{R_q(x)}{Q_m(x)} = \sum_{i_1=1}^{\ell_1} \frac{A_{i_11}}{(x-a_1)^{i_1}} + \cdots + \sum_{i_r=1}^{\ell_r} \frac{A_{i_rr}}{(x-a_r)^{i_r}}$ $+ \sum_{j_1=1}^{m_1} \frac{B_{j_11}x + C_{j_11}}{(x^2 + b_1 x + c_1)^{j_1}} + \cdots + \sum_{j_s=1}^{m_s} \frac{B_{j_ss}x + C_{j_ss}}{(x^2 + b_s x + c_s)^{j_s}}$ where A's, B's, and C's are real numbers. The right hand side can always be integrated!

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