Slide 1

Tricks for trigs

- Review: Recursion formulas.
- Integral of some trigonometric functions.
- integrals of $f : \mathbb{R} \to \mathbb{C}$.

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Reduction formulas are a simple way to write complicated integrals

In the case of the function $\sin(x)$ one has:

$$\int (\sin(x))^n \, dx = -\frac{1}{n}(\sin(x))^{(n-1)} \cos(x) + \frac{(n-1)}{n} \int (\sin(x))^{(n-2)} \, dx.$$

$$\int (\cos(x))^n \, dx = \frac{1}{n}(\cos(x))^{(n-1)} \sin(x) + \frac{(n-1)}{n} \int (\cos(x))^{(n-2)} \, dx.$$

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Tricks for sines and cosines

Consider integrals of the form

$$\int [\sin(x)]^m [\cos(x)]^n dx.$$

If m or n is odd, then the integral can be done by *substitution*, recalling:

$$\sin'(x) = \cos(x), \quad \cos'(x) = -\sin(x),$$

$$\sin^2(x) + \cos^2(x) = 1.$$

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Tricks for sines and cosines

Consider integrals of the form

$$\int [\sin(x)]^m [\cos(x)]^n dx.$$

If both m and n are even, then integral above can be done recalling

$$\sin^2(x) = \frac{1}{2}[1 - \cos(2x)],$$

$$\cos^2(x) = \frac{1}{2}[1 + \cos(2x)].$$

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Tricks for tangents and secants

Consider integrals of the form

$$\int [\tan(x)]^n [\sec(x)]^m dx.$$

If n is odd or if m is even, then *substitution* recalling

$$\tan'(x) = \sec^2(x), \quad \sec'(x) = \tan(x)\,\sec(x),$$

$$\sec^2(x) - \tan^2(x) = 1.$$

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More tricks on sines and cosines

The following integrals can be done with the associated formula:

$$\int \sin(mx)\cos(nx)\,dx \quad \to \quad \sin(\alpha)\cos(\beta) = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)];$$

$$\int \sin(mx)\sin(nx)\,dx \quad \to \quad \sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)];$$

$$\int \cos(mx)\cos(nx)\,dx \quad \to \quad \cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)].$$

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Integration can be generalized to functions $f : I\!R \to \mathbb{C}$

These functions have the form $f(x) = f_1(x) + i\, f_2(x)$, with $f_1(x)$ and $f_2(x)$ real functions.

Examples:

$$\begin{aligned} f(x) &= 2x + i\, \ln(x), \\ f'(x) &= 2 + i\, \frac{1}{x}, \\ g(x) &= \tan(3x) + i\, e^{2x} \\ g'(x) &= 3\sec^2(3x) + 2i\, e^{2x}. \end{aligned}$$

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Riemann sums, integration, and the FTC can be generalized to function $f : I\!R \to \mathbb{C}$

$$\int f(x)dx = \int f_1(x)dx + i \int f_2(x)dx.$$

Example:

$$\int e^{iax}dx = \frac{1}{ia}e^{iax} = -\frac{i}{a}e^{iax}.$$

$$\int [x^2 + i\cos(x)]dx = \frac{1}{3}x^3 + i\sin(x).$$

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Complex tricks for sine and cosine

Euler formulas

$$\begin{aligned} e^{ix} &= \cos(x) + i\sin(x), \\ e^{-ix} &= \cos(x) - i\sin(x), \end{aligned}$$

imply that

$$\begin{aligned} \cos(x) &= \frac{1}{2}[e^{ix} + e^{-ix}], \\ \sin(x) &= \frac{1}{2i}[e^{ix} - e^{-ix}]. \end{aligned}$$

Therefore, integrals in sines and cosines can be transformed into complex integrals for exponentials.

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The subtle art of integration

- Trigonometric substitution.
- Integrals of rational functions
- Examples.

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Here is one more trick with trigs

Integrals of the form

$$\int \sqrt{a^2 - x^2}\, dx, \quad |x| \leq |a|,$$

can be computed with the change of variable:

$$x = a\sin(\theta).$$

Then,

$$\int \sqrt{a^2 - x^2}\, dx = a^2 \int \cos^2(\theta)\, d\theta.$$

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Here are two more tricks that do not always work well, but they are worth to try

Integrals containing factors $\sqrt{a^2 + x^2}$ could be transformed with a change of variables $x = a\tan(\theta)$.

Integrals containing factors $\sqrt{x^2 - a^2}$, with $|x| \geq |a|$, could be transformed with a change of variables $x = a\sec(\theta)$.

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Quotients of polynomials can always be integrated

Definition 1 *A rational function is a function of the form*

$$f(x) = \frac{P_n(x)}{Q_m(x)},$$

where $P_n(x)$ and $Q_m(x)$ are a polynomials of degree n and m, respectively.

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The idea is to learn how to integrate rational functions as the following ones:

$$\int \frac{2x^3 - x^2 + x - 1}{x + 1}\, dx,$$

$$\int \frac{4x^3 - 2x^2 - x - 1}{2x^2 + x - 1}\, dx.$$

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The idea is to rewrite the rational function in a more convenient way

Theorem 1 *Consider a rational function $P_n(x)/Q_m(x)$, with $n > m$. Then, there exist polynomials $S_p(x)$ and $R_q(x)$ of degree $p = n - m$ and $q < m$ such that*

$$\frac{P_n(x)}{Q_m(x)} = S_p(x) + \frac{R_q(x)}{Q_m(x)}.$$

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The idea is to rewrite the rational function in a more convenient way

Theorem 2 *Consider a rational function $R_q(x)/Q_m(x)$ with $q < m$, and where*

$$Q_m(x) = Q_0(x - a_1) \cdots (x - a_m),$$

with $a_1, \cdots, a_m$ are real numbers. Then, there exists real numbers $A_1, \cdots, A_m$ such that

$$\frac{R_q(x)}{Q_m(x)} = \frac{A_1}{(x - a_1)} + \cdots + \frac{A_m}{(x - a_m)}.$$

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How to integrate ratio of polynomials

- Review: Rational functions.
- Integration of rational functions.
- Examples.

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A rational function is a quotient of polynomials

Definition 2 *A rational function is a function of the form*

$$f(x) = \frac{P_n(x)}{Q_m(x)},$$

where $P_n(x)$ and $Q_m(x)$ are a polynomials of degree n and m, respectively.

Quotients of polynomials can always be integrated

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The idea is to rewrite the rational function in a more convenient way

Theorem 3 *Consider a rational function* $P_n(x)/Q_m(x)$*, with* $n > m$*. Then, there exist polynomials* $S_p(x)$ *and* $R_q(x)$ *of degree* $p = n - m$ *and* $q < m$ *such that*

$$\frac{P_n(x)}{Q_m(x)} = S_p(x) + \frac{R_q(x)}{Q_m(x)}.$$

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The problem of integrating a quotient of arbitrary polynomials has been simplified

One needs to study only polynomials of the form

$$\frac{P_n(x)}{Q_m(x)}, \quad 0 \leq n < m.$$

This type of rational functions can be integrated using the following two results.

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Every polynomial can be decomposed into a product of polynomials of degree one and two

Theorem 4 *Every polynomial $Q_m(x)$ with $m \geq 0$ and real coefficients can be decomposed as*

$$Q_m(x) = a(x-a_1)^{\ell_1} \cdots (x-a_r)^{\ell_r} \left(x^2 + b_1\, x + c_1\right)^{m_1} \cdots \left(x^2 + b_s\, x + c_s\right)^{m_s}.$$

with $m = \ell_1 + \cdots + \ell_r + m_1 + \cdots + m_s$.

The $a_1, \cdots, a_r$ are roots of $Q_m(x)$, that is, $Q_m(a_i) = 0$, for $i = 1, \cdots, r$.

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Products can be transformed into sums

Theorem 5 *Consider a rational function $R_q(x)/Q_m(x)$ with $0 \leq q < m$, and where*

$$Q_m(x) = a(x-a_1)^{\ell_1} \cdots (x-a_r)^{\ell_r} \left(x^2 + b_1\, x + c_1\right)^{m_1} \cdots \left(x^2 + b_s\, x + c_s\right)^{m_s}.$$

Then, $R_q(x)/Q_m(x)$ can be written as

$$\frac{R_q(x)}{Q_m(x)} = \sum_{i_1=1}^{\ell_1} \frac{A_{i_1 1}}{(x-a_1)^{i_1}} + \cdots + \sum_{i_r=1}^{\ell_r} \frac{A_{i_r r}}{(x-a_r)^{i_r}}$$

$$+ \sum_{j_1=1}^{m_1} \frac{B_{j_1 1}\, x + C_{j_1 1}}{(x^2 + b_1\, x + c_1)^{j_1}} + \cdots + \sum_{j_s=1}^{m_s} \frac{B_{j_s s}\, x + C_{j_s s}}{(x^2 + b_s\, x + c_s)^{j_s}}$$

where A's, B's, and C's are real numbers.

The right hand side can always be integrated!