Slide 1

$z = a + bi$, powers, roots, and exponentials

- Review: Cartesian and polar representations.
- Powers and roots.
- Exponential and Euler formula.

Slide 2

Complex numbers can be associated with points in a plane

- Cartesian picture: Good for representing addition and real number multiplication.
  (Parallelogram law and stretching.)
- Polar picture: Good for representing the multiplication law.
  (Stretching and rotation.)

Slide 3

The power of a complex number is very easy to compute in the polar representation

Theorem 1 (De Moivre)

$$(r[\cos(\theta) + i\sin(\theta)])^n = r^n[\cos(n\theta) + i\sin(n\theta)].$$

Equivalently:

$$z = r[\cos(\theta) + i\sin(\theta)] \Rightarrow z^n = r^n[\cos(n\theta) + i\sin(n\theta)].$$

Slide 4

Arbitrary powers are easy in polar representation

$$z = a + bi, \quad \Leftrightarrow \quad z = r[\cos(\theta) + i\sin(\theta)],$$

$$r = \sqrt{a^2 + b^2}, \quad \theta = \arctan(b/a).$$

Then,

$$(a + bi)^n = r^n[\cos(n\theta) + i\sin(n\theta)].$$

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Magic at work: There are n solutions to the n-th root of a complex number

(In real numbers there are one or two, for n is odd or even, respectively.)

Theorem 2 *Let* $z = r[\cos(\theta) + i\sin(\theta)]$ *and* $n \geq 1$. *Then, the complex numbers*

$$w_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right)\right]$$

$k = 0, \cdots, n-1$ *satisfy the equation*

$$(w_k)^n = z.$$

Slide 6

Why not to integrate by parts?

- Review: Complex numbers and Euler formula.
- Integration by parts.
- Exercises.
- Recursion formula.

Slide 7

Euler first obtained a formula for the exponential of real numbers

Theorem 3

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n,$$

for all $x \in \mathbb{R}$.

Slide 8

Euler later considered the De Moivre formula

$$[\cos(\theta) + i\sin(\theta)] = \left[\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right]^n.$$

Therefore,

$$[\cos(\theta) + i\sin(\theta)] = \lim_{n\to\infty} \left(1 + \frac{i\theta}{n}\right)^n.$$

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Euler formula is one of the most beautiful formulas we have seen so far

The calculation above *suggests* the following relation:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

In particular, one has Euler formula:

$$e^{i\pi} - 1 = 0.$$

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Why not to integrate by parts?

Theorem 4 (Integration by parts) *If $f(x)$ and $g(x)$ are integrable functions in $[a, b]$, then the following formulas hold,*

$$\begin{aligned} \int f'(x)g(x)\,dx &= f(x)g(x) - \int f(x)g'(x)\,dx, \\ \int_a^b f'(x)g(x)\,dx &= [f(x)g(x)]\big|_a^b - \int_a^b f(x)g'(x)\,dx. \end{aligned}$$

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The proof is based on the product rule and the FTC

Recall that $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$.

Indefinite integral:

$$\begin{aligned} f(x)g(x) &= \int [f(x)g(x)]'\,dx, \\ &= \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx. \end{aligned}$$

Definite integral:

$$\begin{aligned} [f(x)g(x)]\big|_a^b &= \int_a^b [f(x)g(x)]'\,dx, \\ &= \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx. \end{aligned}$$

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Simple examples of integration by parts

Find the following integrals:

$$\begin{aligned} \int \ln(x)\,dx &= x\ln(x) - x, \\ \int xe^x\,dx &= (x-1)e^x, \\ \int x\sin(x)\,dx &= -x\cos(x) + sin(x), \\ \int \frac{1}{x}\ln(x)\,dx &= \frac{1}{2}\ln^2(x). \end{aligned}$$

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Integration by parts is very useful to construct integration tables

Do you know how the following integral was discovered?

$$\int \frac{x^2}{2} e^x \, dx = \left(\frac{x^2}{2} - x + 1 \right) e^x .$$

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Reduction formulas are a simple way to write complicated integrals

In the case of the function $\sin(x)$ one has:

$$\begin{aligned} \int [\sin(x)]^n \, dx &= -\frac{1}{n} [\sin(x)]^{(n-1)} \cos(x) \\ &\quad + \frac{(n-1)}{n} \int [\sin(x)]^{(n-2)} \, dx . \end{aligned}$$