

Integration provides a way to define the average of a continuous function on a closed interval

**Definition 1** The average of a continuous function f(x)in [a, b] is

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$$f_{ave} = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx$$

**Theorem 1** Let m and M be the minimum and maximum values of a continuous function f(x) in [a, b]. Then,

$$m \leq f_{ave} \leq M.$$

The Mean Value Theorem in differential form

**Theorem 2** If f(x) is differentiable in [a, b] with f'(x) continuous in [a, b], then there exists  $c \in (a, b)$  such that

f(b) - f(a) = f'(c) (b - a).

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### The Mean Value Theorem in integral form

**Theorem 3** If f(x) is continuous in [a, b], then there exists  $c \in (a, b)$  such that

$$\int_{a}^{b} f(x) \, dx = f(c) \left(b - a\right)$$

Polar coordinates are useful to describe situations with circular symmetry

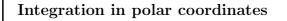
**Definition 2** Let (x, y) be Cartesian coordinates in  $\mathbb{R}^2$ . Then, polar coordinates  $(r, \theta)$  are given by

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$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)$$

The inverse expression is

$$x = r\cos(\theta), \quad y = r\sin(\theta).$$



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- Review of polar coordinates.
- Areas of angular sections.
- Riemann sums and integrals in polar coordinates.
- Examples.

## Review of polar coordinates

**Definition 3** Let (x, y) be Cartesian coordinates in  $\mathbb{R}^2$ . Then, polar coordinates  $(r, \theta)$  are given by

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We review the formula for the area of circular sections

The Greeks proved that for every disk or radius r, circumference  $L_T$ , and area  $A_T$ , the quotients

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$$\frac{L_T}{2r}, \quad \frac{A_T}{r^2}$$

are equal and the same for every disk of any radius r. They called that constant  $\pi$ . Therefore, we have

$$\frac{L_T}{2r} = \frac{A_T}{r^2} = \pi.$$

# Angles are defined as a fraction of the circumference length

Let  $\ell$  be a portion of a circumference in a disk of radius r. Sexagesimal grads are defined as:

$$\hat{\theta}_{\ell} = 360 \frac{\ell}{L_T}, \quad \hat{\theta}_{\ell} \in [0, 360]$$

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Radians are defined as:

$$\theta_{\ell} = \frac{\ell}{r}, \quad \theta_{\ell} \in [0, 2\pi]$$

$$\left(\theta_{\ell} = \frac{\ell}{r} = \frac{L_T}{r} \frac{\ell}{L_T} = 2\pi \frac{\ell}{L_T}\right).$$

therefore,

The area of a circular section of a disk is proportional to the total area

$$A_{\theta_{\ell}} = \frac{\ell}{L_T} A_T,$$

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$$A_{\theta_{\ell}} = \frac{\ell}{r} \frac{r}{L_T} A_T = \theta_{\ell} \frac{1}{2\pi} A_T = \theta_{\ell} \frac{1}{2\pi} \pi r^2$$

so we finally get the area of a circular section to be:

$$A_{\theta} = \frac{1}{2}r^2\theta.$$

### Riemann sums in polar coordinates

Let  $r, \theta$  be polar coordinates in  $\mathbb{R}^2$ .

**Definition 4** Let  $r(\theta)$  be a function defined on a interval  $\theta \in [\theta_a, \theta_b]$ . The integral of  $r(\theta)$  in  $[\theta_a, \theta_b]$  is the number given by

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$$\frac{1}{2} \int_{\theta_a}^{\theta_b} [r(\theta)]^2 d\theta = \frac{1}{2} \lim_{n \to \infty} \sum_{i=0}^{n-1} [r(\theta_i^*)]^2 \Delta\theta,$$

if the limit exists. Given a natural number n we have introduced a partition on  $[\theta_a, \theta_b]$  given by  $\Delta \theta = (\theta_b - \theta_a)/n$ . We denoted  $\theta_i^* = (\theta_i + \theta_{i-1})/2$ , where  $\theta_i = \theta_a + i\Delta \theta$ ,  $i = 0, 1, \dots, n$ .

This choice of the sample point  $\theta_i^*$  is called midpoint rule.

Comparison between integral in Cartesian and in polar coordinates

The area below y = f(x) for  $x \in [a, b]$  is

$$A = \int_{a}^{b} f(x) \, dx.$$

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The area below  $r = g(\theta)$  for  $\theta \in [\theta_a, \theta_b]$  is

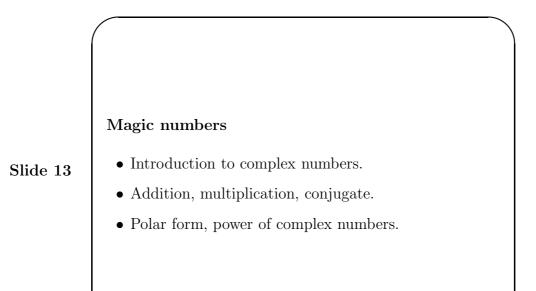
$$A = \frac{1}{2} \int_{\theta_a}^{\theta_b} [g(\theta)]^2 \, d\theta.$$

## Area between two functions in polar coordinates

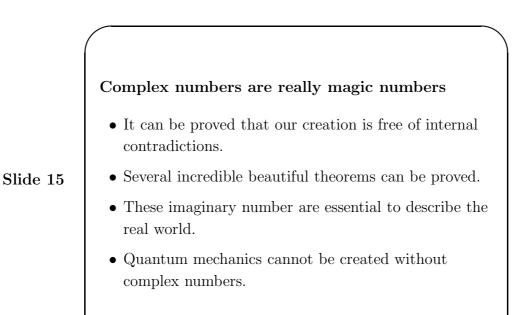
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Given two functions  $r_2(\theta) \ge r_1(\theta)$  the area in between these functions in the interval  $[\theta_a, \theta_b]$  is

$$A = \frac{1}{2} \int_{\theta_a}^{\theta_b} \left( [r_2(\theta)]^2 - [r_1(\theta)]^2 \right) \, d\theta$$



What if we create something?
There is no real number solution of x<sup>2</sup> = -1.
So the symbol √-1 means nothing.
What if
we create a symbol for that nothingness, say,
\$\nabla^2 = \sqrt{-1}\$;
we decide that \$\nabla\$ satisfies every law of addition and multiplication that real numbers do satisfy.
Is the result of that creation, meaningful?



#### Basic operations with complex numbers

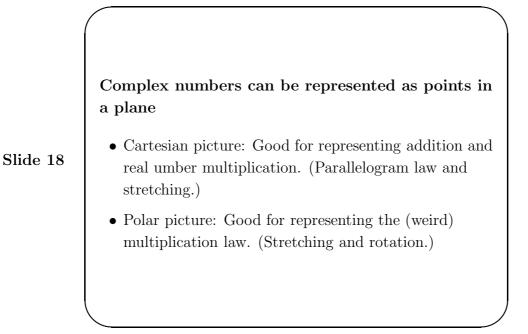
Let z = a + bi, and w = c + di be two complex numbers, a, b, c, d be real numbers. Then we define

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- cz = ca + cbi;
- z + w = (a + c) + (b + d)i;
- zw = (ac bd) + (ad + bc)i;
- $\overline{z} = a bi;$
- $|z|^2 = z\overline{z} = a^2 + b^2$ .

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Notice these strange properties of our creation  $i^0 = 1$ ,  $i^1 = i$ ,  $i^2 = -1$ ,  $i^3 = -i$ ,  $i^4 = 1$ ,  $i^5 = i$ . Also:  $i = -\frac{1}{i}$ .



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