Slide 1

Miscellaneous topics

- Average of a function.
- Integral form of the Mean Value Theorem.
- Polar coordinates.

Slide 2

Integration provides a way to define the average of a continuous function on a closed interval

Definition 1 *The average of a continuous function $f(x)$ in $[a, b]$ is*

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Theorem 1 *Let m and M be the minimum and maximum values of a continuous function $f(x)$ in $[a, b]$. Then,*

$$m \leq f_{ave} \leq M.$$

Slide 3

The Mean Value Theorem in differential form

Theorem 2 *If $f(x)$ is differentiable in $[a, b]$ with $f'(x)$ continuous in $[a, b]$, then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)\,(b - a).$$

The Mean Value Theorem in integral form

Theorem 3 *If $f(x)$ is continuous in $[a, b]$, then there exists $c \in (a, b)$ such that*

$$\int_a^b f(x)\,dx = f(c)\,(b - a).$$

Slide 4

Polar coordinates are useful to describe situations with circular symmetry

Definition 2 *Let (x, y) be Cartesian coordinates in $\mathbb{R}^2$. Then, polar coordinates (r, θ) are given by*

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

The inverse expression is

$$x = r\cos(\theta), \quad y = r\sin(\theta).$$

Slide 5

Integration in polar coordinates

- Review of polar coordinates.
- Areas of angular sections.
- Riemann sums and integrals in polar coordinates.
- Examples.

Slide 6

Review of polar coordinates

Definition 3 *Let (x, y) be Cartesian coordinates in $\mathbb{R}^2$. Then, polar coordinates (r, θ) are given by*

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

The inverse expression is

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Slide 7

We review the formula for the area of circular sections

The Greeks proved that for every disk or radius r, circumference L_T, and area A_T, the quotients

$$\frac{L_T}{2r}, \quad \frac{A_T}{r^2}$$

are equal and the same for every disk of any radius r. They called that constant π. Therefore, we have

$$\frac{L_T}{2r} = \frac{A_T}{r^2} = \pi.$$

Slide 8

Angles are defined as a fraction of the circumference length

Let ℓ be a portion of a circumference in a disk of radius r.

Sexagesimal grads are defined as:

$$\hat{\theta}_\ell = 360\frac{\ell}{L_T}, \quad \hat{\theta}_\ell \in [0, 360].$$

Radians are defined as:

$$\theta_\ell = \frac{\ell}{r}, \quad \theta_\ell \in [0, 2\pi].$$

$$\left(\theta_\ell = \frac{\ell}{r} = \frac{L_T}{r}\frac{\ell}{L_T} = 2\pi\frac{\ell}{L_T}\right).$$

Slide 9

The area of a circular section of a disk is proportional to the total area

$$A_{\theta_\ell} = \frac{\ell}{L_T}\, A_T,$$

therefore,

$$A_{\theta_\ell} = \frac{\ell}{r}\,\frac{r}{L_T}\, A_T = \theta_\ell\,\frac{1}{2\pi}\, A_T = \theta_\ell\,\frac{1}{2\pi}\,\pi r^2,$$

so we finally get the area of a circular section to be:

$$A_\theta = \frac{1}{2} r^2 \theta.$$

Slide 10

Riemann sums in polar coordinates

Let r, θ be polar coordinates in $\mathbb{R}^2$.

Definition 4 *Let $r(\theta)$ be a function defined on a interval $\theta \in [\theta_a, \theta_b]$. The integral of $r(\theta)$ in $[\theta_a, \theta_b]$ is the number given by*

$$\frac{1}{2}\int_{\theta_a}^{\theta_b} [r(\theta)]^2\, d\theta = \frac{1}{2}\lim_{n\to\infty} \sum_{i=0}^{n-1} [r(\theta_i^*)]^2\, \Delta\theta,$$

if the limit exists. Given a natural number n we have introduced a partition on $[\theta_a, \theta_b]$ given by $\Delta\theta = (\theta_b - \theta_a)/n$. We denoted $\theta_i^* = (\theta_i + \theta_{i-1})/2$, where $\theta_i = \theta_a + i\Delta\theta$, $i = 0, 1, \cdots, n$.

This choice of the sample point θ_i^* is called midpoint rule.

Slide 11

Comparison between integral in Cartesian and in polar coordinates

The area below $y = f(x)$ for $x \in [a, b]$ is

$$A = \int_a^b f(x)\, dx.$$

The area below $r = g(\theta)$ for $\theta \in [\theta_a, \theta_b]$ is

$$A = \frac{1}{2} \int_{\theta_a}^{\theta_b} [g(\theta)]^2\, d\theta.$$

Slide 12

Area between two functions in polar coordinates

Given two functions $r_2(\theta) \geq r_1(\theta)$ the area in between these functions in the interval $[\theta_a, \theta_b]$ is

$$A = \frac{1}{2} \int_{\theta_a}^{\theta_b} \left([r_2(\theta)]^2 - [r_1(\theta)]^2 \right) d\theta.$$

Slide 13

Magic numbers

- Introduction to complex numbers.
- Addition, multiplication, conjugate.
- Polar form, power of complex numbers.

Slide 14

What if we create something?

There is no real number solution of $x^2 = -1$.

So the symbol $\sqrt{-1}$ means nothing.

What if

- we create a symbol for that nothingness, say,

$$\heartsuit^2 = \sqrt{-1};$$

- we decide that $\heartsuit$ satisfies every law of addition and multiplication that real numbers do satisfy.

Is the result of that creation, meaningful?

Slide 15

Complex numbers are really magic numbers

- It can be proved that our creation is free of internal contradictions.
- Several incredible beautiful theorems can be proved.
- These imaginary number are essential to describe the real world.
- Quantum mechanics cannot be created without complex numbers.

Slide 16

Basic operations with complex numbers

Let $z = a + bi$, and $w = c + di$ be two complex numbers, a, b, c, d be real numbers. Then we define

- $cz = ca + cbi$;
- $z + w = (a + c) + (b + d)i$;
- $zw = (ac - bd) + (ad + bc)i$;
- $\overline{z} = a - bi$;
- $|z|^2 = z\overline{z} = a^2 + b^2$.

Slide 17

Notice these strange properties of our creation

$$i^0 = 1, \quad i^1 = i, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i.$$

Also:

$$i = -\frac{1}{i}.$$

Slide 18

Complex numbers can be represented as points in a plane

- Cartesian picture: Good for representing addition and real umber multiplication. (Parallelogram law and stretching.)
- Polar picture: Good for representing the (weird) multiplication law. (Stretching and rotation.)