Slide 1

Review on Integration (Secs. 5.1 - 5.3)

- Remarks on the course.
- Review: Sec. 5.1-5.3
  - Origins of Calculus.
  - Riemann Sums.
  - New functions from old ones.

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A mathematical description of motion motivated the creation of Calculus

Problem of Motion:

- Given $x(t)$ find $v(t)$ $\leftrightarrow$ Differential Calculus.
- Given $v(t)$ find $x(t)$ $\leftrightarrow$ Integral Calculus.

Derivatives and integrals are operations on functions.

One is the inverse of the other. This is the content of the *Fundamental theorem of Calculus.*

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An integral is a sum of infinite many terms

Definition 1 (Integral of a function) *Let $f(x)$ be a function defined on a interval $x \in [a, b]$. The integral of $f(x)$ in $[a, b]$ is the number given by*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i^*)\,\Delta x,$$

if the limit exists. Given a natural number n we have introduced a partition on $[a, b]$ given by $\Delta x = (b - a)/n$. We denoted $x_i^* = (x_i + x_{i-1})/2$, where $x_i = a + i\Delta x$, $i = 0, 1, \cdots, n$. This choice of the sample point x_i^* is called midpoint rule.

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An integral is a sum of infinite many terms

Continuous functions are integrable. The sum of infinite many terms is finite.

Theorem 1 *If $f(x)$ is continuous in $[a, b]$, then*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n, \quad \textit{exists.}$$

Notation: $\int_a^b f(x)\,dx$ is called the definite integral of $f(x)$ from a to b.

Notice: $\int_a^b f(x)\,dx$ is a number.

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Properties deduced from the definition

$$
\begin{aligned}
\int_a^b f(x)\,dx &= -\int_b^a f(x)\,dx; \\
\int_a^a f(x)\,dx &= 0; \\
\int_a^b c\,dx &= c(b-a); \\
\int_a^b (f \pm g)\,dx &= \int_a^b f\,dx \pm \int_a^b g\,dx;
\end{aligned}
$$

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More properties deduced from the definition

$$
\begin{aligned}
\int_a^b c\,f(x)\,dx &= c\int_a^b f(x)\,dx; \\
\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^a f(x)\,dx; \\
f \geq 0 &\Rightarrow \int_a^b f\,dx \geq 0; \\
f \geq g &\Rightarrow \int_a^b f\,dx \geq \int_a^b g\,x; \\
m \leq f \leq M &\Rightarrow m(b-a) \leq \int_a^b f\,dx \leq M(b-a).
\end{aligned}
$$

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Integration can be used to define new functions from old ones

Theorem 2 *If $f(x)$ is continuous in $[a, b]$, then*

$$F(x) = \int_a^x f(s)\, ds, \quad x \in [a, b],$$

is a continuous functions and $F(a) = 0$.

Examples:

$$\ln(x) = \int_1^x \frac{1}{s}\, ds, \qquad x^2 = \int_0^x 2s\, ds.$$

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Fundamental Theorem of Calculus

- Review: New function using integration.
- Fundamental Theorem of Calculus. (Sec. 5.3)
- Integration tables. (Sec. 5.4)

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Integration can be used to define new functions from old ones

Theorem 3 *If $f(x)$ is continuous in $[a, b]$, then*

$$F(x) = \int_a^x f(s)\,ds, \quad x \in [a, b],$$

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Examples:

$$\ln(x) = \int_1^x \frac{1}{s}\,ds, \qquad x^2 = \int_0^x 2s\,ds.$$

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Derivation and integration are operations on functions, and they are inverse of each other

Theorem 4 (Fundamental Theorem of Calculus) *If $f(x)$ is continuous in $[a, b]$ and c is any constant, then*

$$F(x) = \int_a^x f(s)\,ds + c, \quad x \in [a, b],$$

is differentiable and

$$F'(x) = f(x).$$

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The FTC justifies the name antiderivation for integration

Definition 2 *The function $F(x)$ given by*

$$F(x) = \int_a^x f(s)\,ds + c$$

for any constant c is called the antiderivative of $f(x)$.

The antiderivative of a given function is not unique.
Two antiderivatives of the same function can differ by a constant.

Notation: $\int f\,dx$ also denotes an antiderivative of f.

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Here are two simple reformulations of the FTC

Corollary 1 *If $f(x)$ is continuous in $[a, b]$, then*

$$\int_a^b f(s)\,ds = F(b) - F(a),$$

where $F(x)$ is any antiderivative of $f(x)$.

Corollary 2 *If $F(x)$ is differentiable in $[a, b]$, then*

$$\int_a^b F'(s)\,ds = F(b) - F(a).$$

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The insight from the FTC can be used to construct integration tables

$$\begin{aligned}
\int a dx &= ax + c, \\
\int x^n dx &= \frac{1}{n+1} x^{n+1} + c, \quad n \neq -1, \\
\int \frac{1}{x}\, dx &= \ln(x) + c, \\
\int e^x dx &= e^x + c.
\end{aligned}$$

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The insight from the FTC can be used to construct integration tables

$$\begin{aligned}
\int \sin(x)\, dx &= -\cos(x) + c, \\
\int \cos(x)\, dx &= \sin(x) + c, \\
\int \frac{1}{1+x^2}\, dx &= \arctan(x) + c, \\
\int \frac{1}{\sqrt{1-x^2}}\, dx &= \arcsin(s) + c.
\end{aligned}$$

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FTC and the chain rule

$$F(x) = \int_a^x f(t)\,dt + c, \quad F(a) = c, \quad F'(x) = f(x).$$

$$\mathbb{F}(x) = F(g(x)), \quad \Rightarrow \quad \mathbb{F}'(x) = F'(g(x))\,g'(x).$$

$$\mathbb{F}(x) = \int_a^{g(x)} f(t)dt + c, \quad \Rightarrow \quad \mathbb{F}'(x) = f(g(x))\,g'(x).$$

Example:

$$f(x) = \int_1^{\sin(x)} \ln(t)\,dt, \quad \Rightarrow \quad f'(x) = \ln(\sin(x))\cos(x).$$

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Substitution rule

- Review: Derivation $\leftrightarrow$ Integration
  are inverse operations.
- Chain rule $\leftrightarrow$ Substitution rule.
- Examples.

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Derivation and integration are inverse operations

$$F(x) = \int_a^x f(s)\,ds + c \quad \Rightarrow \quad F(a) = c, \quad F'(x) = f(x).$$

In other words,

$$F(x) = \int f(x)\,dx \quad \Rightarrow \quad F'(x) = f(x).$$

In still other words,

$$\int F'(x)\,dx = F(x).$$

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Derivation tables ⇒ Integration tables

$$(x^n)' = nx^{n-1} \quad \Rightarrow \quad \int nx^{n-1}\,dx = x^n + c,$$

$$[\sin(x)]' = \cos(x) \quad \Rightarrow \quad \int \cos(x)\,dx = \sin(x) + c.$$

Derivation rules ⇒ Integration rules

Chain rule ⇒ Substitution rule.

Product rule ⇒ Integration by parts.

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Chain rule provides a way to compute some integrals

Find the primitive (antiderivative) of $2x\cos(x^2)$.

$$\begin{aligned} \int \cos(x^2) 2x\, dx &= \int [\sin(x^2)]'\, dx, \\ &= \sin(x^2) + c. \end{aligned}$$

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Chain rule provides a way to compute some integrals

Find the primitive (antiderivative) of $\cos(x)/\sin(x)$.

$$\begin{aligned} \int \frac{1}{\sin(x)} \cos(x)\, dx &= \int [\ln(\sin(x))]'\, dx, \\ &= \ln(\sin(x)) + c. \end{aligned}$$

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Here is the general case of this inverse form of chain rule

Find the primitive of $f(g(x))\, g'(x)$ knowing that the primitive of $f(x)$ is $F(x)$, that is, $F'(x) = f(x)$.

$$\begin{aligned} \int f(g(x))\, g'(x)\, dx &= \int F'(g(x))\, g'(x)\, dx, \\ &= \int [F(g(x))]'\, dx, \\ &= F(g(x)) + c. \end{aligned}$$

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The substitution rule is a technique to use the inverse form of the chain rule efficiently

Recall, $F'(x) = f(x)$, then

$$\int f(g(x))\, g'(x)\, dx = \int F'(g(x))\, g'(x)\, dx.$$

Introduce $u = g(x)$, then denote $du = g'(x)\, dx$.
Substitute these expressions in the right hand side above:

$$\begin{aligned} \int f(g(x))\, g'(x)\, dx &= \int F'(u)\, du, \\ &= F(u) + c, \\ &= F(g(x)) + c. \end{aligned}$$

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The substitution rule on definite integrals changes the limits of integration

Theorem 5 (Change of variable) *Let $g(x)$ be differentiable in $[a, b]$ with $g'(x)$ continuous in $[a, b]$. Let $f(u)$ be continuous for $u = g(x)$ and $x \in [a, b]$. Then,*

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

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Proof of the change of variable theorem

Let $F(x)$ be a primitive of $f(x)$, so $F'(x) = f(x)$, and $F(d) - F(c) = \int_c^d f(u)\, du$.

$$\begin{aligned} \int_a^b f(g(x))\, g'(x)\, dx &= \int_a^b F'(g(x))\, g'(x)\, dx, \\ &= \int_a^b [F(g(x))]'\, dx, \\ &= F(g(b)) - F(g(a)), \\ &= \int_{g(a)}^{g(b)} f(u)\, du. \end{aligned}$$