Math 20B Supplement
Linked to Stewart, Edition 5

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Supplement to Appendix G and Chapters 7 and 9 of Stewart Calculus Edition 5

Contents

1   Complex Exponentials: For Appendix G Stewart Edition 5

This material is a supplement to Appendix G of Stewart. You should read the appendix, except maybe the last section on complex exponentials, *before* this material.

How should we define e^{a+bi} where a and b are real numbers? In other words, what is e^z when z is a complex number? We would like the nice properties of the exponential to still be true. Probably, some of the most basic properties are that for any complex numbers z and w we have

$$e^{z+w} = e^z \, e^w \qquad \text{and} \qquad \frac{d}{dx} \, e^{wx} = we^{wx}. \tag{1.1}$$

It turns out that the following definition produces a function with these properties.

> **Definition of complex exponential:** $e^{a+bi} = e^a(\cos b + i \sin b) = e^a \cos b + ie^a \sin b$

In particular, for any real number x, Euler's formula holds true:

$$e^{ix} = \cos x + i \sin x. \tag{1.2}$$

We now prove the first key property in (1.1).

Theorem 1.1 *If z and w are complex numbers, then*

$$e^{z+w} = e^z e^w.$$

Proof.

$z = a + ib$ and $w = h + ik$

$$\begin{aligned}
e^z e^w &= e^a(\cos b + i \sin b)e^h(\cos k + i \sin k) \\
&= e^a e^h([\cos b \cos k - \sin b \sin k] + i[\cos b \sin k + \sin b \cos k]) \\
&= e^{a+h}[\cos(b+k) + i \sin(b+k)] \\
&= e^{[a+h+i(b+k)]} \\
&= e^{z+w}
\end{aligned}$$

∎

We leave checking the second property to the exercises. For those who are interested there is an appendix, Section 6, which discusses what we mean by derivative of a function of complex variables and explains how to obtain the second property as well.

It's easy to get formulas for the trigonometric functions in terms of the exponential. Look at Euler's formula (1.2) with x replaced by $-x$:

$$e^{-ix} = \cos x - i \sin x.$$

We now have two equations in $\cos x$ and $\sin x$, namely

$$\begin{aligned}
\cos x + i \sin x &= e^{ix} \\
\cos x - i \sin x &= e^{-ix}.
\end{aligned}$$

Adding and dividing by 2 gives us $\cos x$ whereas subtracting and dividing by $2i$ gives us $\sin x$:

$$\boxed{\textbf{Exponential form of sine and cosine:}\quad \cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}}$$

Setting $x = z = a + bi$ gives formulas for the sine and cosine of complex numbers. We can do a variety of things with these formula. Here are some we will not pursue:

- Since the other trigonometric functions are rational functions of sine and cosine, this gives us formulas for all the trigonometric functions.

- The hyperbolic and trigonometric functions are related:

$$\cos x = \cosh(ix) \text{ and } i \sin x = \sinh(ix).$$

1.1 Complex Exponentials Yield trigonometric Identities

The exponential formulas we just derived, together with $e^{z+w} = e^z e^w$ imply the identities

$$\sin^2 \alpha + \cos^2 \alpha = 1$$
$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$
$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.$$

These three identities are the basis for deriving trigonometric identities. Hence we can derive trigonometric identities by using the exponential formulas and $e^{z+w} = e^z e^w$. We now illustrate this with some examples.

Example 1.2 Show that $\cos^2 x + \sin^2 x = 1$. Indeed, we have

$$
\begin{aligned}
\left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 + \left(\frac{e^{ix} + e^{-ix}}{2i}\right)^2 &= \frac{1}{4}\left[(e^{ix})^2 + 2 + (e^{-ix})^2 + \frac{(e^{ix})^2 - 2 + (e^{-ix})^2}{i^2}\right] \\
&= \frac{1}{4}\left[2 + 2\right] = 1,
\end{aligned}
$$

wherein we have used the fact that $i^2 = -1$.   ∎

Example 1.3

$$
\begin{aligned}
\sin 2x &= \frac{e^{i2x} - e^{-i2x}}{2i} = \frac{1}{2i}\left[(e^{ix})^2 - (e^{-ix})^2\right] \\
&= 2\frac{[e^{ix} - e^{-ix}]}{2i}\frac{[e^{ix} + e^{-ix}]}{2} \\
&= 2 \sin x \cos x
\end{aligned}
$$

∎

1.2   Exercises

1. Use the relationship between the sine, cosine and exponential functions to express $\cos^3 x$ as a sum of sines and cosines.

2. Show that $e^{\pi i} + 1 = 0$. This uses several basic concepts in mathematics, such as π, e, addition, multiplication and exponentiation of complex numbers in one compact equation.

3. What are the cartesian coordinates x and y of the complex number $x + iy = e^{2+3i}$?

4. Use the fact that

$$\frac{d}{dx}\left[\cos(bx) + i\sin(bx)\right] = b[-\sin(bx) + i\cos(bx)]$$

and the product rule to prove that

$$\frac{d}{dx}\left[e^{(a+ib)x}\right] = (a + ib)e^{(a+ib)x},$$

This is the key differentiation property for complex exponentials.

2   Integration of Functions which Take Complex Values: For Chapter 7.2 Stewart Edition 5

This supplements Chapter 7.2 of Stewart Ed. 5.

Now we turn to the issue of integrating functions which take complex values. Of course this is bound up with what we mean by antiderivatives of complex functions. A function, such as $f(x) = (1+2i)x + i3x^2$, may have complex values but the variable x is only allowed to take on real values and we only define definite integrals for this type of functions. In this case nothing differs from what we already learned about integrals of real valued functions.

- The Riemann sum definition of an integral still applies.

- The Fundamental Theorem of Calculus is still true.

- The properties of integrals, including substitution and integration by parts still work.

For example,

$$\int_0^2 ((1+2i)x + 3ix^2)\, dx = \int_0^2 x\, dx + 2i \int_0^2 x\, dx + 3i \int_0^2 x^2\, dx = \frac{(1+2i)x^2}{2} + ix^3 \Big]_0^2$$
$$= (1+2i)2 + 8i = 2 + 12i.$$

On the other hand, we can't evaluate $\int_0^1 (x+i)^{-1}\, dx$ right now. Why is that? We would expect to write $\int (x+i)^{-1}\, dx = \ln(x+i) + C$ and use the Fundamental Theorem of Calculus, but this has no meaning[1] because we only know how to compute logarithms of positive numbers.

2.1   Integrating Products of Sines, Cosines and Exponentials

In Section 7.2 products of sines and cosines were integrated using trigonometric identities. There are other ways to do this now that we have complex exponentials.

Examples will make this clearer.

Example 2.1 Let's integrate $8\cos 3x \sin x$.

$$8\cos 3x \sin x = 8\left(\frac{e^{3ix} + e^{-3ix}}{2}\right)\left(\frac{e^{ix} - e^{-ix}}{2i}\right)$$
$$= \frac{2}{i}\left(e^{4ix} + e^{-2ix} - e^{2ix} - e^{-4ix}\right).$$

It is not difficult to integrate this, namely,

$$\int 8\cos 3x \sin x\, dx = \frac{2}{i}\int e^{4ix} + e^{-2ix} - e^{2ix} - e^{-4ix}\, dx$$
$$= \frac{2}{i}\left[\frac{e^{4ix}}{4i} - \frac{e^{-2ix}}{2i} - \frac{e^{2ix}}{2i} + \frac{e^{-4ix}}{4i}\right]$$

---

[1] Some of you might suggest that we write $\ln|x+i|$ instead of $\ln(x+i)$. This does not work. Since $|x+i| = \sqrt{x^2+1}$, the function $f(x) = \ln|x+i|$ only takes on real values when x is real. Its derivative cannot be the complex number $(x+i)^{-1}$ since $(f(x+h) - f(x))/h$ is real.

Sort this by powers of $e^{\pm x}$ to get

$$\frac{2}{i}\left[\frac{e^{4ix}}{4i} + \frac{e^{-4ix}}{4i} - \frac{e^{-2ix}}{2i} - \frac{e^{2ix}}{2i}\right]$$

$$= -\cos 4x + 2\cos 2x$$

∎

Example 2.2 Let's integrate $e^{2x}\sin x$. Problems like this were solved in Section 7.1 by using integration by parts twice. Here is another way. Using the formula for sine and integrating we have

$$\int e^{2x}\sin x\, dx \;\;=\;\; \frac{1}{2i}\int e^{2x}(e^{ix} - e^{-ix})\,dx \;\;=\;\; \frac{1}{2i}\int (e^{(2+i)x} - e^{(2-i)x})\,dx$$

$$=\;\; \frac{1}{2i}\left(\frac{e^{(2+i)x}}{2+i} - \frac{e^{(2-i)x}}{2-i}\right) + C$$

$$=\;\; \frac{-ie^{2x}}{2}\left(\frac{e^{ix}(2-i)}{5} - \frac{e^{-ix}(2+i)}{5}\right) + C$$

Sort by powers of $e^{\pm x}$ to get

$$\frac{-ie^{2x}}{10}\left(2(e^{ix} - e^{-ix}) - i(e^{+ix} + e^{-ix})\right) + C$$

$$=\; \frac{-e^{2x}}{10}\left(-4\sin x + 2\cos x\right) + C$$

∎

This method works for integrals of products of sines, cosines and exponentials, and often for quotients of them, (though this requires more advanced methods, such as partial fractions). The advantage of using complex exponentials is that it takes the guess out of computing such integrals. The method, however, could be messier than the one presented in the book, though it is often simpler. We also point out that Example 2.1 went beyond those illustrated in the book.

2.2 Exercises

Compute the following integrals using complex exponentials.

1. $\displaystyle\int_{-\pi}^{\pi} 7\sin(5x)\cos(3x)\, dx$

2. $\displaystyle\int e^{i7x}\cos(2x)\, dx$

3. $\displaystyle\int \cos^2(x)\, e^{-3x}\, dx$

4. $\displaystyle\int \cos^3(x)\cos(7x)\, dx$

5. $\displaystyle\int \sin^2(x)e^{\sqrt{5}x}\, dx$

6. $\displaystyle\int x\cos^3(x)\, dx$

7. $\displaystyle\int \sin^3(x)\cos(10x)\, dx$

3 The Fundamental Theorem of Algebra: For Chapter 7.4 Stewart Edition 5

A polynomial p of degree n is a function of the form

$$Q(x) = Q_0 + Q_1 x + Q_2 x^2 + \cdots + Q_n x^n. \tag{3.1}$$

where the coefficients Q_j can be either real or complex numbers. The following is a basic fact which is hard to prove (and we shall not attempt a proof here).

> **Fundamental Theorem of Algebra:** Any nonconstant polynomial can be factored as a product of linear factors with complex coefficients times a constant. Linear factors are of the form $x - \beta$.

This tells us that we can factor a polynomial of degree n into a product of n linear factors. For example,

- $3x^2 + 2x - 1 = 3(x - \frac{1}{3})(x + 1)$ ($n = 2$ here),

- $x^3 - 8 = (x - 2)(x + \alpha)(x + \overline{\alpha})$ where $\alpha = 1 \pm i\sqrt{3}$ ($n = 3$ here),

- $(x^2 + 1)^2 = (x + i)^2 (x - i)^2$ ($n = 4$ here).

3.1 Zeroes and their multiplicity

Notice that a very important feature of the factorization is:

> *Each factor $x - \beta$ of p corresponds to a number β which is a* **zero of the polynomial** *p, namely,*
> $$Q(\beta) = 0.$$

To see this just consider the factorization of p evaluated at β, namely

$$Q(\beta) = c(\beta - \beta_1)(\beta - \beta_2)\dots(\beta - \beta_n).$$

This is equal to zero if and only if one of the factors is 0; say the *jth* factor is zero, which gives $\beta - \beta_j = 0$. Thus $\beta = \beta_j$ for some j.

For some polynomials a factor $x - \beta_j$ will appear more than once, for example, in

$$Q(x) = 7(x - 2)^5 (x - 3)(x - 8)^2$$

the $x - 2$ factor appears 5 times, the $x - 3$ factor appears once, the $x - 8$ factor appears twice. The jargon for this is

$$2 \text{ is a zero of } p \text{ of multiplicity } 5$$

$$3 \text{ is a zero of } p \text{ of multiplicity } 1$$

$$8 \text{ is a zero of } p \text{ of multiplicity } 2.$$

The general form for a factored polynomial is

$$Q(x) = k(x - \beta_1)^{m_1}(x - \beta_2)^{m_2}\dots(x - \beta_\ell)^{m_\ell} \tag{3.2}$$

where β_j is called **a zero of Q of multiplicity** m_j and k is a constant.

3.2   Real Coefficients

All polynomials which you see in math 20B have real coefficients. So it is useful to give a version of the Fundamental Theorem of Algebra all numbers in the factoring are real.

Fundamental Theorem of Algebra: real factors   Any nonconstant polynomial p with real coefficients can be factored as a product of linear factors and quadratic factors all having real coefficients, that is,

$$Q(x) = c(x - r_1)^{m_1} \cdots (x - r_\ell)^{m_\ell} \, (x^2 + b_1 x + c_1)^{n_1} \cdots (x^2 + b_k x + c_k)^{n_k}. \qquad (3.3)$$

Later we shall study rational functions $f = \frac{P}{Q}$. The partial fraction expansions in Ch 7.4 of Stewart are based on this version of the Fundamental Theorem of Algebra. Thus, if we allow complex numbers, partial fractions can be done with only linear factors. When we only allowed real numbers as coefficients of the factors, we obtained both linear and quadratic factors, as does Stewart.

Proof: A useful fact is:

If all the coefficients Q_j of the polynomial Q are real numbers, then

$$Q(\beta) = 0 \quad \text{implies} \quad Q(\overline{\beta}) = 0.$$

To see this think of x as a real number. Suppose $(x - \alpha)^k$ is a factor of Q, then $(x - \overline{\alpha})^k$ is also a factor:

(a) Since $(x - \alpha)^k$ is a factor of $Q(x)$, we have $Q(x) = (x - \alpha)^k r(x)$ for some polynomial $r(x)$.

(b) Taking complex conjugates, $\overline{Q(x)} = (x - \overline{\alpha})^k \overline{r(x)}$.

(c) Since $Q(x)$ has real coefficients, $\overline{Q(x)} = Q(x)$ and so by (b), $(x - \overline{\alpha})^k$ is a factor of $Q(x)$.

This and the Fundamental Theorem of Algebra (with complex factors) implies a polynomial Q with real coefficients has a factorization

$$Q(x) = (x - \beta_1)(x - \overline{\beta}_1) \cdots (x - \beta_k)(x - \overline{\beta}_k)(x - r_1) \cdots (x - r_\ell) \qquad (3.4)$$

or equivalently

$$Q(x) = (x^2 + b_1 x + c_1) \cdots (x^2 + b_k x + c_k)(x - r_1) \cdots (x - r_\ell) \qquad (3.5)$$

where $b_1, \ldots, b_k$ and $c_1, \cdots, c_k$ and $r_1, \ldots, r_\ell$ are real numbers. In fact you can check that $b_j = 2\mathrm{Re}\,\beta_j$ and $c_j = |\beta_j|^2$. ∎

The advantage of the first version of the Fundamental Theorem Algebra is that all terms are linear in x and the disadvantage is that some of them may contain numbers β_j which are not real. The advantage of the second version of the Fundamental Theorem Algebra is that all numbers in the factoring are real.

3.3   Rational Functions and Poles

The quotient of two polynomials $\frac{P}{Q}$ is called a **rational function**. For a rational function f we call any number β for which $|f(x)|$ is not bounded as $x \to \beta$ a **pole** of f. For example,

$$f(x) = \frac{x^7}{(x-1)^2(x-9)^3}$$

has poles at 1, 9 and ∞. You might think calling ∞ a pole peculiar, but $\lim\limits_{x\to\infty} |f(x)| = \infty$ as the definition requires. Poles have multiplicity; in this case

> 1 is a pole of p of multiplicity 2
>
> 9 is a pole of p of multiplicity 3
>
> ∞ is a pole of p of multiplicity 2, since $f(x) \sim x^2$ as $x \to \infty$.

A rational function is called **proper** if $\lim_{x\to\infty} |f(x)| = 0$.

The growth rate of f near a high multiplicity pole exceeds that of f near a low multiplicity pole.

3.4   Exercises

1. Expand $Q = (x-2)(x-3)(x-2+1)$ in the form (3.1).

2. Show that if P is a polynomial and $P(5) = 0$, then $\frac{P(x)}{x-5}$ is a polynomial.

3. (a)How many poles does the rational function $r(x) = \frac{3}{6+x+5x^3}$ have?
   Does it have a "pole at ∞"?
   (b)What are the pole locations and their multiplicities for $r(x) = \frac{3-2x}{(x-2)(x^2+5x+7)}$?

4. The following is the simplest mathematical model used for a building hit by an earthquake. If the bottom of the building is displaced horizontally from rest a distance $b(t)$ at time t, then the roof of the building is displaced from vertical by a distance $r(t)$. The issue is to describe the relationship between b and r in a simple way. Fortunately, there is a rational function $\mathcal{T}(s)$ called the **transfer function** of the building with the property that when b is a pure sine wave

$$b(t) = \sin wt$$

   at frequency $\frac{w}{2\pi}$, then r is a sine wave of *the same frequency*[2] and with amplitude $|\mathcal{T}(iw)|$.

   While earthquakes are not pure sine waves, they can be modelled by combinations of sine waves.

   If

$$\mathcal{T}(s) = \frac{s^2}{(s+3i+.01)(s-3i+.01)(s+7i+.1)(s-7i+.1)},$$

   then at approximately what frequency does the building shake the most, the second most?

---

[2]r has the form $r(t) = |\mathcal{T}(iw)|\sin(wt + \psi(iw))$

5. Electric circuits behave similarly and are typically described by their transfer function $\mathcal{T}$. If $c(t)$, a sinusoidal current of frequency $w/2\pi$ is imposed, and $v(t)$ is the voltage one measures it is a sine wave of the same frequency with amplitude $|\mathcal{T}(iw)|$.

If
$$\mathcal{T}(s) = \frac{1}{(s+3i+.01)(s-3i+.01)} + \frac{2}{s-10},$$

then approximately how much accuracy do we lose in predicting the amplitude for our output with the simpler mathematical model

$$\widetilde{\mathcal{T}}(s) = \frac{1}{(s+3i+.01)(s-3i+.01)}$$

When a sine wave at frequency $\frac{w}{2\pi}$ is put in?

Hint: You may use the fact that $\left| |\widetilde{\mathcal{T}}(s)| - |\mathcal{T}(s)| \right| \leq |\widetilde{\mathcal{T}}(s) - \mathcal{T}(s)|$, even though we have not proved it.

4   Partial Fraction Expansions (PFE): For Chapter 7.4 Stewart Edition 5

The partial fraction expansion (PFE) of a rational function can be found by the method in the text. In this section we point out that there are easier methods for computing the constants involved. Later in Subsection 4.4 we give some intuition about the form of a partial fraction expansion. You might find that this helps you remember the form of the PFE.

4.1   A Shortcut when there are no Repeated Factors

You especially save a lot of time when there are no repeated factors in the denominator. We'll tell you the general principle and then do some specific examples.

Suppose that $\alpha_1, \ldots, \alpha_n$ are all distinct. Suppose also that the degree of $P(x)$ is less than n. Then

$$\frac{P(x)}{(x - \alpha_1) \cdots (x - \alpha_n)} = \frac{C_1}{x - \alpha_1} + \cdots + \frac{C_n}{x - \alpha_n}, \tag{4.1}$$

where the constants $C_1, \ldots, C_n$ we need to be determined to find the partial fraction expansion.

- *Multiply both sides of (4.1) by $x - \alpha_j$ and then set $x = -\alpha_j$.*
  The left side is some number $\mathcal{N}$.

- *On the right side, we are left with only C_j,*
  because all the other terms have a factor of $x - \alpha_j$ which is 0 when $x = \alpha_j$.

- *Thus $\mathcal{N} = C_j$*

Now for some illustrations.

Example 4.1 (Partial fractions with all factors linear: none repeated, proper)
This is CASE I type in Stewart Ed 5.
Let's expand $f(x) := \frac{x^2+2}{(x-1)(x+2)(x+3)}$ by partial fractions.

$$f(x) = \frac{x^2 + 2}{(x - 1)(x + 2)(x + 3)} = \frac{C_1}{x - 1} + \frac{C_2}{x + 2} + \frac{C_3}{x + 3}$$

Multiply by $x - 1$ to eliminate the pole at $x = 1$ and get

$$(x - 1)f(x) = \frac{x^2 + 2}{(x + 2)(x + 3)} = C_1 + \frac{C_2(x - 1)}{x + 2} + \frac{C_3(x - 1)}{x + 3}.$$

Set $x = 1$ and get

$$\frac{1 + 2}{(1 + 2)(1 + 3)} = C_1$$

and so $C_1 = \frac{1}{4}$. Similarly,

$$C_2 = (x + 2)f(x) \Big]_{x=-2} = \frac{x^2 + 2}{(x - 1)(x + 3)} \Big]_{x=-2} = \frac{4 + 2}{(-3)1} = -2$$

and

$$C_3 = (x+3)f(x)\Big]_{x=-3} = \frac{x^2+2}{(x-1)(x+2)}\Big]_{x=-3} = \frac{9+2}{(-4)(-1)} = \frac{11}{4}.$$

We conclude

$$f(x) = \frac{x^2+2}{(x-1)(x+2)(x+3)} = \frac{1}{4(x-1)} - \frac{2}{(x+2)} + \frac{11}{4(x+3)}.$$

A cultural aside is that the numbers C_1, C_2, C_3 are often (though not in Stewart) are called *the residues* of the poles at $1, -2, -3$, many of you will see them later in your career under that name.

If we wish to find the antiderivatives of f from this we immediately get

$$\int f(x)dx = \frac{1}{4}ln|x-1| + 2\,ln|x+2| + \frac{11}{4}ln|x+3| + K$$

∎

4.2   The Difficulty with Repeated Factors

These are of CASE II type in Stewart Ed. 5.
Let us apply the previous method to

$$f(x) = \frac{1}{(x-1)^2(x-3)}$$

whose partial fraction expansion we know (by Stewart's book) has the form

$$f(x) = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{x-3}. \tag{4.2}$$

We can find C quickly from

$$C = (x-3)f(x)\Big]_{x=3} = \frac{1}{(3-1)^2} = \frac{1}{4}$$

and A from

$$A = (x-1)^2 f(x)\Big]_{x=1} = \frac{1}{1-3} = -\frac{1}{2}.$$

However, B does not succumb to this technique; you must use other means to find it. What we have gotten from our method is just the coefficients of the "highest terms" at each pole.

To find B many ways will do. For example, the one in Stewart will do and we have made it go much faster by finding A and C. Another way to find the missing number B is to plug in one value of x, say $x=0$ and get

$$\frac{1}{(-1)^2(-3)} = f(0) = -\frac{1}{2} - B - \frac{1}{3}\,\frac{1}{4}$$

$$-B = \left[\frac{-1}{3} + \frac{1}{2} + \frac{1}{12}\right]\cdot$$

$$B = \frac{+1}{12}.$$

To summarize

$$f(x) = \frac{-1}{2(x-1)^2} + \frac{1}{12(x-1)} + \frac{1}{4(x-3)}.$$

The antiderivatives of f are

$$\int f(x)dx = \frac{1}{2(x-1)} + \frac{1}{12}ln|x-1| + \frac{1}{4}ln|x-3| + K.$$

4.3 Every Rational Function has a Partial Fraction Expansion

Now we mention a pleasant fact.

Theorem 4.2 *Every rational function* $f = \frac{P}{Q}$ *has a partial fraction expansion.*

The core of the reason is the Fundamental Theorem of Algebra, which can be used to factor Q as in formula (3.2). This produces,

$$f(x) = \frac{P(x)}{(x-\beta_1)^{m_1}(x-\beta_2)^{m_2}(x-\beta_\ell)^{m_\ell}}.$$

If the numerator and denominator polynomials defining f have real coefficients, then f can always be written

$$f(x) = \frac{P(x)}{(x-r_1)^{m_1}\cdots(x-r_\ell)^{m_\ell}\,(x^2+b_1x+c_1)^{n_1}\cdots(x^2+b_kx+c_k)^{n_1}}$$

with all coefficients in the factors real numbers. This is the factoring behind the various cases treated in Stewart Chapter 7.4 Ed 5. One then needs to write out the appropriate form for the PFE and then identify the coefficients as has been explained in Stewart Ed.5 Chapter 7.4 and in these notes for cases where all factors are linear (even with high multiplicity) and where there is a multiplicity one quadratic factor, Case IV Stewart Ed. 5.

4.4 The Form of the Partial Fraction Expansion

Here is one way to look at the form of the PFE of a rational function f. We just give the rough idea which may be too vague to be very helpful.

Recall that a high multiplicity pole has a "faster growth rate" that a lower multiplicity pole. Thus it can "overshadow" the lower multiplicity role.

Example 4.3 The function $f(x) = \frac{1}{(x-1)^2(x-3)}$ has a multiplicity 2 pole at 1 and a multiplicity 1 pole at 3. Thus the PFE has the form

$$f(x) = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{x-3}.$$

The role of the multiplicity 1 pole at 3 is obvious. Let us turn to giving intuition behind the multiplicity 2 pole at 1. Its "strength is"

$$\frac{1}{(x-1)^2}\frac{1}{(1-3)} = \frac{-1}{2}\frac{1}{(x-1)^2},$$

but when we subtract this pole from f we get

$$
\begin{aligned}
e(x) &= f(x) - \frac{-1}{2}\frac{1}{(x-1)^2} = \frac{1}{(x-1)^2}\left[\frac{1}{x-3}+\frac{1}{2}\right] \\
e(x) &= \frac{1}{(x-1)(x-3)}
\end{aligned}
$$

which still has a pole at 1, though now it is a pole of multiplicity 1. Thus we must include a first order pole at 1 in the PFE. That is why both $\frac{A}{(x-1)^2}$ and $\frac{B}{(x-1)}$ must be included. ∎

Similar intuition tells us that $f(x) = \frac{x^7}{(x-1)^2(x-9)^3}$ has a PFE of the form

$$
f(x) = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{(x-9)^3} + \frac{D}{(x-9)^2} + \frac{E}{x-9} + Fx^2 + Gx + H,
$$

since it has poles at 1 of multiplicity 2, at 9 of multiplicity 3 and at ∞ of multiplicity 2.

4.5   More Examples: Non-proper Rational Functions and Quadratic Factors

Example 4.4 (Partial fractions with a pole at infinity: all linear factors and none repeated.)
This is CASE I type in Stewart Ed 5.
Let's expand $f(x) := \frac{x^3+2}{(x-1)(x+2)}$ by partial fractions. Clearly, f has a pole at 1,-2 and ∞ all of multiplicity one. The form of the PFE is

$$
f(x) = \frac{x^3+2}{(x-1)(x+2)} = \frac{C_1}{x-1} + \frac{C_2}{x+2} + Ax + B \tag{4.3}
$$

where $Ax+B$ is included to pick up the pole at ∞. Indeed, $Ax+B$ is the simplest rational function containing the general multiplicity one pole structure at infinity. Beware you must include B.

Now solve for the $A, B, C's$. Multiply by $x-1$ to eliminate the pole at $x=1$ and get

$$
(x-1)f(x)\Big]_{x=1} = \frac{x^3+2}{(x+2)}\Big]_{x=1} = C_1.
$$

That is,

$$
C_1 = \frac{1+2}{(1+2)} = 1
$$

Similarly,

$$
C_2 = (x+2)f(x)\Big]_{x=-2} = \frac{x^3+2}{(x-1)}\Big]_{x=-2} = \frac{-8+2}{(-3)} = 2.
$$

Finding A is easy since it is the "highest order term" at infinity. First observe

$$
\lim_{x\to\infty}\frac{f(x)}{x} = A
$$

Then

$$
A = \lim_{x\to\infty}\frac{f(x)}{x} = \lim_{x\to\infty}\frac{x^2+2/x}{(x-1)(x+2)} = \lim_{x\to\infty}\frac{x^2}{x^2} = 1
$$

Now we must only find B. As usual there are many ways to do this. For example, plug $x = 0$ into (4.3) and get

$$\frac{2}{(-1)2} = f(0) = \frac{C_1}{-1} + \frac{C_2}{2} + +B = \frac{1}{-1} + \frac{2}{2} + +B = B$$

Thus $B = -1$.   ∎

Example 4.5 (Partial fractions with no repeated factors: an irreducible quadratic term, proper) This is CASE III in Stewart Ed 5.
Let's find a PFE of $\frac{x+1}{x^3+x}$. Note that $f(x) = \frac{x+1}{x^3+x}$ has two natural forms of partial fraction expansions corresponding to whether we factor the denominator $x^3 + x$ in the form (3.5) with real coefficients or (3.4) with complex coefficients. Stewart Ch. 7.4 Ed. 5 uses (3.5) so we emphasize and recommend that one, namely

$$f(x) = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}.$$

We proceed like Stewart, but save a little time with

$$A = xf(x)\Big]_{x=0} = \frac{1}{1} = 1$$

Next multiply by $x(x^2 + 1)$ to get $x + 1 = x(x^2 + 1)f(x) = x^2 + 1 + x(Bx + C)$. Cancel ones and divide by x, to get

$$1 = x + Bx + C.$$

Set $x = 0$ to get $C = 1$ and so $B = -1$. Thus the PFE is

$$f(x) = \frac{x + 1}{x^3 + x} = \frac{1}{x} + \frac{-x + 1}{x^2 + 1}.$$

If we want antiderivatives this gives

$$\int f(x)dx = ln|x| - \frac{1}{2}ln|x^2 + 1| + \arctan x + K$$

This solves the problem completely.

While students probably will not use the (3.4) form of expansion, for the sake of the curious, (the less curious can skip this) we show how it is done.

$$f(x) = \frac{x + 1}{(x^3 + x)} = \frac{x + 1}{x(x - i)(x + i)} = \frac{C_1}{x} + \frac{C_2}{x - i} + \frac{C_3}{x + i}.$$

Since $x = x - 0$,

$$xf(x)\Big]_{x=0} = C_1 = \frac{1}{(-i)i} = 1.$$

Also

$$(x - i)f(x)\Big]_{x=i} = C_2 = \frac{i + 1}{i(2i)} = \frac{-1 - i}{2} \quad \text{and} \quad (x + i)f(x)\Big]_{x=-i} = C_3 = \frac{-i + 1}{(-i)(-2i)} = \frac{-1 + i}{2}.$$

Note that $C_3 = \bar{C}_2$ and we can get the first PFE from this PFE by

$$f(x) = \frac{1}{x} + \frac{C_2}{x-i} + \frac{C_3}{x+i} = \frac{1}{x} + \frac{C_2(x+i) + C_3(x-i)}{x^2+1} = \frac{1}{x} + \frac{2\mathrm{Re}C_2 x + (-2)\mathrm{Im}C_2}{x^2+1}$$

$$f(x) = \frac{1}{x} + \frac{-x+1}{x^2+1}.$$

which is what we got before.   ∎

We did not do higher multiplicity quadratic factors here, Case IV Stewart Ed. 5.

4.6   Exercises

1. K   Use partial fraction techniques to solve
   Exercise K for K equal 17 through 38 in Section 7.4 Stewart Edition 5.

2. Find the partial fraction expansion of $\frac{2x+1}{(x-1)^2(x+2)}$.

3. Given $f(x) = \frac{3}{(x-1)(x-2)^2}$. What value of A makes $f(x) - \frac{A}{x-1}$ have its only pole located at 2?

4. Find the partial fraction expansion of $\frac{x^3+2}{x(x^2+1)(x^2+4)}$.

5. Find the partial fraction expansion of $\frac{x^3+2}{x(x^2-1)(x^2-4)}$.

6. Consider the PFE of r in (4.2). We claim that

$$\frac{d}{dx}\left[(x-1)^2 f(x)\right]\Big|_{x=1}$$

is either A B, or C in the partial fraction expansion.
(a) Which one is it? (b) Does such a formula hold for any rational function with a second order pole? (Justify why). (c) Find a similar formula for a rational function with a third order pole.

5 Improving on Euler's Method: For Chapter 9.2 Stewart Ed. 5

This supplements Chapter 9.2 Stewart Edition 5. Among its prerequisites is Chapter 7.7 on numerical integration.

Suppose we are given the differential equation $y' = F(x, y)$ with initial condition $y(x_0) = y_0$. Euler's method, discussed in Section 9.2, produces a sequence of approximations $y_1, y_2, \ldots$ to $y(x_1), y(x_2), \ldots$ where $x_n = x_0 + nh$ are equally spaced points.

This is almost the left endpoint approximation in numerical integration (Chapter 7 of Stewart Ed. 5). To see this, suppose that we have an approximation y_{n-1} for $y(x_{n-1})$, and that we want an approximation for $y(x_n)$. Integrate $y' = F(x, y)$ from x_{n-1} to x_n and use the left endpoint approximation:

$$y(x_n) - y(x_{n-1}) = \int_{x_{n-1}}^{x_n} F(x, y)\, dx \approx hF(x_{n-1}, y(x_{n-1})).$$

Now we have a problem that did not arise in numerical integration: We don't know $y(x_{n-1})$. What can we do? We replace $y(x_{n-1})$ with the approximation y_{n-1} to obtain

$$y(x_n) - y_{n-1} \approx hF(x_{n-1}, y_{n-1}).$$

Rearranging and calling the approximation to $y(x_n)$ thus obtained y_n we have Euler's method:

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}). \tag{5.1}$$

We know that the left endpoint approximation is a poor way to estimate integrals and that the Trapezoidal Rule is better. Can we use it here? Adapting the argument that led to (5.1) for use with the Trapezoidal Rule gives us

$$y_n = y_{n-1} + \frac{h}{2}\Big(F(x_{n-1}, y_{n-1}) + F(x_n, y_n)\Big). \tag{5.2}$$

You should carry out the steps. Unfortunately, (5.2) can't be used: We need y_n on the right side in order to compute it on the left!

Here is a way around this problem: First, use (5.1) to estimate ("predict") the value of y_n and call this prediction y_n^*. Second, use y_n^* in place of y_n in the right side of (5.2) to obtain a better estimate, called the "correction". The formulas are

$$\text{(predictor)} \quad y_n^* = y_{n-1} + hF(x_{n-1}, y_{n-1}) \tag{5.3}$$

$$\text{(corrector)} \quad y_n = y_{n-1} + \frac{h}{2}\Big(F(x_{n-1}, y_{n-1}) + F(x_n, y_n^*)\Big).$$

This is an example of a *predictor-corrector* method for differential equations. Here are results for Example 9.2.3, the differential equation $y' = x + y$ with initial condition $y(0) = 1$:

step size	$y(1)$ by (5.1)	$y(1)$ by (5.3)
0.50	2.500000	3.281250
0.20	2.976640	3.405416
0.10	3.187485	3.428162
0.05	3.306595	3.434382
0.02	3.383176	3.436207
0.01	3.409628	3.436474

The correct value is 3.436564, so (5.3) is much better than Euler's method for this problem.

5.1   Exercises

1. Write down a predictor-corrector method based on Simpson's Rule for numerical integration. Hint: a bit tricky is that we consider not two, but three grid points x_{n-2}, x_{n-1}, x_n and assume we know f_{n-2} and f_{n-1}. The problem for you is to give an algorithm for producing f_n.

6   Appendix: Differentiation of Complex Functions

Suppose we have a function $f(z)$ whose values are complex numbers and whose variable z may also be a complex number. We can define limits and derivatives as Stewart did for real numbers. Just as for real numbers, we say the complex numbers z and w are "close" if $|z - w|$ is small, where $|z - w|$ is the absolute value of a complex number.[3]

- We say that $\lim_{z \to \alpha} f(z) = L$ if, for every real number $\epsilon > 0$ there is a corresponding real number $\delta > 0$ such that

$$|f(z) - L| < \epsilon \qquad \text{whenever} \qquad 0 < |z - \alpha| < \delta.$$

- The derivative is defined by $f'(\alpha) = \lim_{z \to \alpha} \dfrac{f(z) - f(\alpha)}{z - \alpha}$.

Our variables will usually be real numbers, in which case z and α are real numbers. Nevertheless the value of a function can still be a complex number because our functions contain complex constants; for example, $f(x) = (1 + 2i)x + 3ix^2$.

Since our definitions are the same, the formulas for the derivative of the sum, product, quotient and composition of functions still hold. Of course, before we can begin to calculate the derivative of a particular function, we have to know how to calculate the function.

What functions can we calculate? Of course, we still have all the functions that we studied with real numbers. So far, all we know how to do with complex numbers is basic arithmetic. Thus we can differentiate a function like $f(x) = \dfrac{1 + ix}{x^2 + 2i}$ or a function like $g(x) = \sqrt{1 + i}\, e^x$ since $f(x)$ involves only the basic arithmetic operations and $g(x)$ involves a (complex) constant times a real function, e^x, that we know how to differentiate. On the other hand, we cannot differentiate a function like e^{ix} because we don't even know how to calculate them.

6.1   Deriving the Formula for e^z Using Differentiation

Two questions left dangling in Section 1 were

- How did you come up with the definition of complex exponential?

- How do you know it satisfies the simple differential equation properties?

We consider each of these in turn.

In Appendix G Stewart uses Taylor series to come up with a formula for e^{a+bi}. Since you haven't studied Taylor series yet, we take a different approach.

From the first of formula in (1.1) with $\alpha = a$ and $\beta = b$, e^{a+bi} should equal $e^a\, e^{bi}$. Thus we only need to know how to compute e^{bi} when b is a real number.

Think of b as a variable and write $f(x) = e^{xi} = e^{ix}$. By the second property in (1.1) with $\alpha = i$, we have $f'(x) = if(x)$ and $f''(x) = if'(x) = i^2 f(x) = -f(x)$. It may not seem like we're getting anywhere, but we are!

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[3]The definitions are nearly copies of Stewart Sections 2.4 and 2.8. We have used z and α instead of x and a to emphasize the fact that they are complex numbers and have called attention to the fact that δ and ϵ are real numbers.

Look at the equation $f''(x) = -f(x)$. There's not a complex number in sight, so let's forget about them for a moment. Do you know of any real functions $f(x)$ with $f''(x) = -f(x)$? Yes. Two such functions are $\cos x$ and $\sin x$. In fact,

$$\text{If } f(x) = A \cos x + B \sin x, \text{ then } f''(x) = -f(x).$$

We need constants (probably complex) so that it's reasonable to let $e^{ix} = A \cos x + B \sin x$. How can we find A and B? When $x = 0$, $e^{ix} = e^0 = 1$. Since

$$A \cos x + B \sin x = A \cos 0 + B \sin 0 = A,$$

we want $A = 1$. We can get B by looking at $(e^{ix})'$ at $x = 0$. You should check that this gives $B = i$. (Remember that we want the derivative of e^{ix} to equal ie^{ix}.) Thus we get

$$\boxed{\textbf{Euler's formula:} \quad e^{ix} = \cos x + i \sin x}$$

Putting it all together we finally have our definition for e^{a+bi}.

We still need to verify that our definition for e^z satisfies (1.1). The verification that $e^{\alpha+\beta} = e^\alpha e^\beta$ is left as an exercise. We will prove that $(e^z)' = e^z$ for complex numbers. Then, by the Chain Rule, $(e^{\alpha x})' = (e^{\alpha x})(\alpha x)' = \alpha e^{\alpha x}$, which is what we wanted to prove.

Example 6.1 (A proof that $(e^z)' = e^z$)

By the definition of derivative and the fact that $e^{\alpha+\beta} = e^\alpha e^\beta$ with $\alpha = z$ and $\beta = w$, we have

$$(e^z)' = \lim_{w \to 0} \frac{e^{z+w} - e^z}{w} = \lim_{w \to 0} \frac{e^z(e^w - 1)}{w} = e^z \lim_{w \to 0} \frac{e^w - 1}{w}.$$

Let $w = x + iy$ where x and y are small real numbers. Then, using the definition of complex exponential, we get

$$\frac{e^w - 1}{w} = \frac{e^x(\cos y + i \sin y) - 1}{x + iy}.$$

Since x and y are small, we can use linear approximations [4] for e^x, $\cos y$ and $\sin y$, namely $1 + x$, 1 and y. (The approximation $\cos y \approx 1$ comes from $(\cos y)' = 0$ at $y = 0$.) Thus $\frac{(e^w-1)}{w}$ is approximately equal to

$$\frac{(1+x)(1+iy) - 1}{x + iy} = \frac{(1+x) + i(1+x)y - 1}{x + iy} = \frac{(x+iy) + ixy}{x + iy} = 1 + \frac{ixy}{x + iy}$$

When x and y are very small, their product is much smaller than either one of them. Thus $\lim_{w \to 0} \frac{ixy}{x+iy} = 0$ and so $\lim_{w \to 0} \frac{(e^w-1)}{w} = 1$. This shows that $(e^z)' = e^z$. ∎

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[4] Linear approximations are discussed in Section 3.11 of Stewart.