Name: ______________________ **Section Number:** ______

TA Name: ______________________ **Section Time:** ______

Math 20B.
Midterm Exam 2
May 26, 2006

No calculators or any other devices are allowed on this exam.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
Read each question carefully. If any question is not clear, ask for clarification.
Answer each question completely, and show all your work.

1. Evaluate the following integrals,
   (a) (8 points) $\displaystyle\int x \ln\left(\frac{x}{2}\right)\, dx$,
   (b) (8 points) $\displaystyle\int (x^2 - 3x)e^x\, dx$.

(a) The integral above is a particular case of $\displaystyle\int x\ln(ax)\,dx$ for $a = 1/2$. Integrating by parts,

$$\begin{aligned}
\int x\ln(ax)\,dx &= \left[\frac{x^2}{2}\ln(ax) - \int \frac{x^2}{2}\frac{1}{ax}dx\right],\\
&= \frac{x^2}{2}\ln(ax) - \frac{1}{2a}\int x\,dx,\\
&= \frac{x^2}{2}\ln(ax) - \frac{x^2}{4a} + c,\\
&= \frac{x^2}{2}\left[\ln(ax) - \frac{1}{2a}\right] + c.
\end{aligned}$$

Therefore, the solution is $\displaystyle\int x\ln\left(\frac{x}{2}\right)\,dx = \frac{x^2}{2}\left[\ln\left(\frac{x}{2}\right) - 1\right] + c.$

(b) The integral above is a particular case of $\displaystyle\int (x^2 - ax)e^x\,dx$ for $a = 3$. Integrating twice by parts,

$$\begin{aligned}
\int (x^2 - ax)e^x\,dx &= (x^2 - ax)e^x - \int (2x - a)e^x\,dx,\\
&= (x^2 - ax)e^x - \left[(2x - a)e^x - \int 2e^x\,dx\right],\\
&= (x^2 - ax)e^x - (2x - a)e^x + 2e^x + c,\\
&= \left[x^2 - (2 + a)x + (2 + a)\right]e^x + c.
\end{aligned}$$

Therefore, the solution is $\displaystyle\int (x^2 - 3x)e^x\,dx = \left[x^2 - 5x + 5\right]e^x + c.$

2. (10 points) Evaluate the integral $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$.

The integral above is a particular case of $\displaystyle\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx$ for $a=4$. Use the trigonometric substitution $x = a\sin(\theta)$, then $dx = a\cos(\theta)\,d\theta$ and $\sqrt{a^2-x^2} = a\cos(\theta)$.

$$\begin{aligned}
\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx &= \int a^2\sin^2(\theta)\frac{1}{a\cos(\theta)}a\cos(\theta)\,d\theta,\\
&= a^2\int \sin^2(\theta)\,d\theta,\\
&= \frac{a^2}{2}\int [1-\cos(2\theta)]\,d\theta,\\
&= \frac{a^2}{2}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + c,\\
&= \frac{a^2}{2}\left[\theta - \sin(\theta)\cos(\theta)\right] + c,\\
&= \frac{a^2}{2}\left[\arcsin\left(\frac{x}{a}\right) - \frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}\right] + c,\\
&= \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) - \frac{x}{2}\sqrt{a^2-x^2} + c,
\end{aligned}$$

Therefore, the solution is $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx = 8\arcsin\left(\frac{x}{4}\right) - \frac{x}{2}\sqrt{16-x^2} + c.$

3. (15 points) Use a partial fraction decomposition to evaluate the integral

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\, dx.$$

First divide the numerator by the denominator. The result is:

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

Notice that the polynomial in the denominator has two roots, 3 and -1, so $x^2 - 2x - 3 = (x+1)(x-3)$. Then,

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{5x - 3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3},$$

which implies the following equations for A and B,

$$A + B = 5, \quad A - 3B = -3, \quad \Rightarrow \quad A = 2, \quad B = 3.$$

Therefore,

$$\begin{aligned} \int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\, dx &= \int 2x\, dx + \int \frac{2}{x+1}\, dx + \int \frac{3}{x-3}\, dx, \\ &= x^2 + 2\ln(|x+1|) + 3\ln(|x-3|) + c. \end{aligned}$$

4. (a) (5 points) Does the integral $\displaystyle\int_1^\infty \frac{\sin^2(3x)}{x^4}\,dx$ converge?

(b) (10 points) Evaluate the integral $\displaystyle\int_e^\infty \frac{dx}{x\ln^2(x)}$.

(a) Notice that $0 \leq \sin^2(3x) \leq 1$, therefore,

$$\int_1^\infty \frac{\sin^2(3x)}{x^4}\,dx \leq \int_1^\infty \frac{1}{x^4}\,dx = \lim_{x\to\infty}\int_1^x t^{-4}\,dt = \lim_{x\to\infty}\left[\frac{t^{-3}}{-3}\Big|_1^x\right] = \lim_{x\to\infty}\left[\frac{1}{3} - \frac{x^{-3}}{3}\right] = \frac{1}{3}.$$

Therefore,

$$0 \leq \int_1^\infty \frac{\sin^2(3x)}{x^4}\,dx \leq \frac{1}{3},$$

and then, the integral in (a) converges by the comparison theorem.

(b)

$$\begin{aligned}
\int_e^\infty \frac{dx}{x\ln^2(x)} &= \lim_{x\to\infty}\int_e^x \frac{dt}{t\ln^2(t)}, \qquad u = \ln(t), \quad du = \frac{1}{t}\,dt,\\
&= \lim_{x\to\infty}\int_1^{\ln(x)} u^{-2}\,du,\\
&= \lim_{x\to\infty}\left(-u^{-1}\Big|_1^{\ln(x)}\right),\\
&= \lim_{x\to\infty}\left(1 - \frac{1}{\ln(x)}\right),\\
&= 1.
\end{aligned}$$