Name: ______________________ **Section Number:** ______

TA Name: ______________________ **Section Time:** ______

Math 20B.
Midterm Exam 1
April 28, 2006

No calculators or any other devices are allowed on this exam.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
Read each question carefully. If any question is not clear, ask for clarification.
Answer each question completely, and show all of your work.

1. (10 points) Evaluate the following integrals.

(a) $\displaystyle\int_0^{\pi/3} \frac{\sin(\theta)}{\cos^2(\theta)}\, d\theta$

(b) $\displaystyle\int \frac{1+x}{1+x^2}\, dx$

(a) Substitute $u = \cos(\theta)$, so $du = -\sin(\theta)\, d\theta$, and $u(0) = 1$, $u(\pi/3) = 1/2$. Then

$$\int_0^{\pi/3} \frac{\sin(\theta)}{\cos^2(\theta)}\, d\theta = \int_1^{1/2} -\frac{du}{u^2} = \int_{1/2}^{1} u^{-2}\, du = \frac{1}{(-1)}\left(u^{-1}\Big|_{1/2}^{1}\right) = -(1-2) = 1.$$

(b)

$$\int \frac{1+x}{1+x^2}\, dx = \int \frac{1}{1+x^2}\, dx + \int \frac{x}{1+x^2}\, dx$$

Integrate the first term directly, and in the second term do the substitution $u = 1 + x^2$, with $du = 2x\, dx$, then

$$\begin{aligned}\int \frac{1+x}{1+x^2}\, dx &= \arctan(x) + \int \frac{1}{2}\frac{du}{u}, \\ &= \arctan(x) + \frac{1}{2}\ln(u) + c, \\ &= \arctan(x) + \ln(\sqrt{1+x^2}) + c.\end{aligned}$$

#	Score
1	
2	
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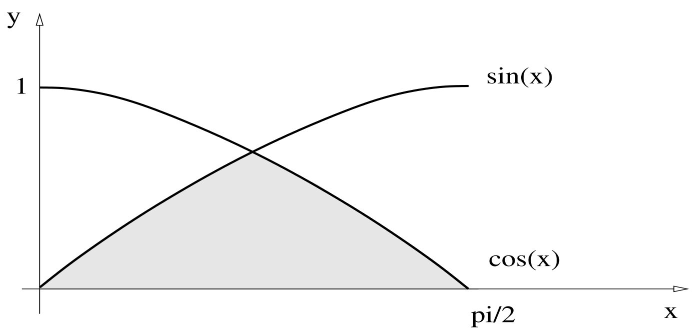
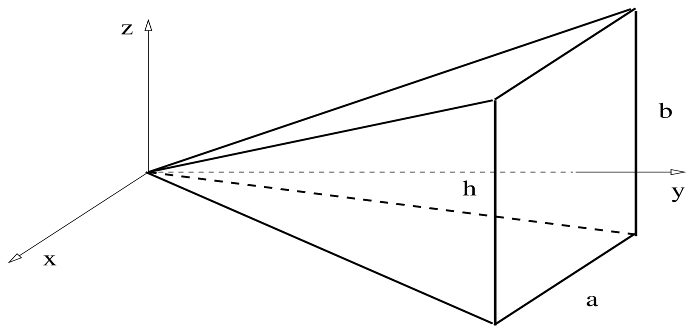
2. (10 points) Find the area of the shaded region



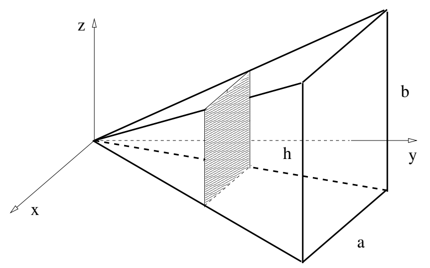
Split the integration region in two intervals, $[0, \pi/4]$ and $[\pi/4, \pi/2]$. Then, the area A of the shaded region is

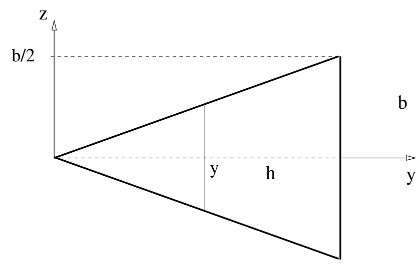
$$
\begin{aligned}
A &= \int_0^{\pi/4} \sin(x)\, dx + \int_{\pi/4}^{\pi/2} \cos(x)\, dx, \\
&= -\cos(x)\big|_0^{\pi/4} + \sin(x)\big|_{\pi/4}^{\pi/2}, \\
&= -\left(\frac{\sqrt{2}}{2} - 1\right) + \left(1 - \frac{\sqrt{2}}{2}\right), \\
&= 2 - \sqrt{2}.
\end{aligned}
$$

3. (10 points) Find the volume of a pyramid with rectangular base of sides a and b, and height h.



The idea is to find the area of rectangular regions perpendicular to the y-axis, as function of y.





This area is given by $A(y) = 2z(y)\, 2x(y)$.

Concentrate in the $x = 0$ plane, then the top of the pyramid is a line in the zy-plane, passing through the origin, so the function $z(y)$ is given by

$$z(y) = \frac{(b/2)}{h}\, y.$$

Analogously, the $x(y)$ function is a line given by

$$x(y) = \frac{(a/2)}{h}\, y.$$

Then

$$A(y) = \frac{ab}{h^2}\, y^2.$$

Then,

$$V = \int_0^h \frac{ab}{h^2}\, y^2\, dy = \frac{ab}{h^2}\, \frac{1}{3} h^3 \quad \Rightarrow \quad V = \frac{1}{3} abh.$$

4. (8 points) Compute both $(1+i)^8$ and $(1+i)^{10}$.

$$z = 1 + i \quad \Rightarrow \quad \begin{cases} r = \sqrt{1+1} = \sqrt{2}, \\ \tan(\theta) = 1 \;\Rightarrow\; \theta = \pi/4. \end{cases}$$

$$z = 2^{1/2}[\cos(\pi/4) + i\sin(\pi/4)] \quad \Rightarrow$$

$$\Rightarrow \quad z^8 = 2^{8/2}[\cos(8\pi/4) + i\sin(8\pi/4)] = 2^4[\cos(2\pi) + i\sin(2\pi)] = 16.$$

$$(1+i)^8 = 16.$$

$$z = 2^{1/2}[\cos(\pi/4) + i\sin(\pi/4)] \quad \Rightarrow$$

$$\Rightarrow \quad z^{10} = 2^{10/2}[\cos(10\pi/4) + i\sin(10\pi/4)] = 2^5[\cos(5\pi/2) + i\sin(5\pi/2)] \quad \Rightarrow$$

$$\Rightarrow \quad z^{10} = 32[\cos(\pi/2) + i\sin(\pi/2)] = 32\,i.$$

$$(1+i)^{10} = 32\,i.$$