Print Name: ______________________ **Section Number:** ______

TA Name: ______________________ **Section Time:** ______

Math 20B.
Final Exam
June 12, 2006

No calculators or any other devices are allowed on this exam.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
Read each question carefully. If any question is not clear, ask for clarification.
Answer each question completely, and show all your work.

1. Evaluate the integrals:

   (a) (8 points) $\displaystyle\int x^2 \cos(x)\, dx$;

   (b) (8 points) $\displaystyle\int \frac{\sin\bigl(\ln(2x)\bigr)}{x}\, dx$.

   Show all your work.

(a)

$$\begin{aligned}\int x^2 \cos(x)\, dx &= x^2 \sin(x) - \int 2x \sin(x)\, dx,\\ &= x^2 \sin(x) - \left[-2x\cos(x) - \int 2\bigl(-\cos(x)\bigr)\, dx\right],\\ &= x^2 \sin(x) + 2x\cos(x) - 2\int \cos(x)\, dx,\\ &= x^2 \sin(x) + 2x\cos(x) - 2\int \sin(x) + c,\end{aligned}$$

$$\boxed{\int x^2 \cos(x)\, dx = x^2 \sin(x) + 2x\cos(x) - 2\int \sin(x) + c}.$$

(b) Substitute $u = \ln(2x)$, then $du = dx/x$, so

$$\begin{aligned}\int \frac{\sin\bigl(\ln(2x)\bigr)}{x}\, dx &= \int \sin(u)\, du,\\ &= -\cos(u) + c,\end{aligned}$$

$$\boxed{\int \frac{\sin\bigl(\ln(2x)\bigr)}{x}\, dx = -\cos\bigl[\ln(2x)\bigr] + c}.$$

2. (8 points) Use complex exponentials to compute the integral $\int e^{2x}\cos^2(x)\,dx$. You may leave the result in exponential form.

$$\begin{aligned}\int e^{2x}\cos^2(x)\,dx &= \int e^{2x}\left[\frac{e^{ix}+e^{-ix}}{2}\right]^2 dx,\\ &= \frac{1}{4}\int e^{2x}\left[e^{2ix}+e^{-2ix}+2\right]dx,\\ &= \frac{1}{4}\int \left[e^{2(1+i)x}+e^{2(1-i)x}+2e^{2x}\right]dx,\\ &= \frac{1}{4}\left[\frac{e^{2(1+i)x}}{2(1+i)}+\frac{e^{2(1-i)x}}{2(1-i)}+e^{2x}\right]+c,\end{aligned}$$

$$\boxed{\int e^{2x}\cos^2(x)\,dx = \frac{1}{4}\left[\frac{e^{2(1+i)x}}{2(1+i)}+\frac{e^{2(1-i)x}}{2(1-i)}+e^{2x}\right]+c}.$$

3. (10 points) Evaluate the integral $\int \frac{2x-1}{x^2-3x+2}\,dx$.

The roots of x^2-3x+2 are $x=2$ and $x-1$, then $x^2-3x+2=(x-2)(x-1)$.

$$\frac{2x-1}{x^2-3x+2} = \frac{A}{x-2} + \frac{B}{x-1} = \frac{(x-1)A+(x-2)B}{(x-2)(x-1)}$$

$$2x-1 = (A+B)x-(A+2B) \quad \Rightarrow \quad A+B=2, \quad A+2B=1,$$

The solutions are $A=3$, $B=-1$.

$$\begin{aligned}\int \frac{2x-1}{x^2-3x+2}\,dx &= \int \frac{3}{x-2}\,dx - \int \frac{1}{x-1}\,dx,\\ &= 3\ln(|x-2|)-\ln(|x-1|)+c.\end{aligned}$$

$$\boxed{\int \frac{2x-1}{x^2-3x+2}\,dx = 3\ln(|x-2|)-\ln(|x-1|)+c}.$$

4. (10 points) Use the comparison Theorem to decide whether $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$ converges or diverges. If the integral converges, then compute its value.
Show your work.

$$x^2 \leqslant 1+x^2 \quad \Rightarrow \quad \frac{1}{1+x^2} \leqslant \frac{1}{x^2} \quad \Rightarrow \quad \int_1^\infty \frac{dx}{1+x^2} \leqslant \int_1^\infty \frac{dx}{x^2}.$$

$$\int_1^\infty \frac{dx}{x^2} = \lim_{t\to\infty} \int_1^t x^{-2}\,dx = \lim_{t\to\infty} \left[1 - \frac{1}{t}\right] = 1.$$

$$0 \leqslant \int_1^\infty \frac{dx}{1+x^2} \leqslant 1, \qquad \text{the integral converges.}$$

$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{t\to\infty} \int_1^t \frac{dx}{1+x^2} = \lim_{t\to\infty} \big[\arctan(t) - \arctan(1)\big] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

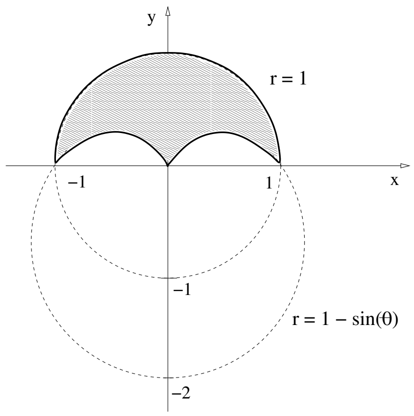
$$\boxed{\int_1^\infty \frac{dx}{1+x^2} = \frac{\pi}{4}}.$$

5. (10 points) Find the volume of the solid of revolution generated by a 2π-rotation around the x-axis of the curve $y = \dfrac{2}{x}$, where $1 \leqslant x \leqslant 4$.

$$V = \pi \int_1^4 \big[y(x)\big]^2 \, dx = \pi \int_1^4 \frac{4}{x^2} \, dx = 4\pi \left(-\frac{1}{x} \right) \bigg|_1^4 = 4\pi \left(1 - \frac{1}{4} \right) = 3\pi.$$

$$\boxed{V = 3\pi}.$$

6. (10 points) Find the area of the region that lies inside the circle $r = 1$ and outside the curve $r = 1 - \sin(\theta)$.



Let $r_1(\theta) = 1$ and $r_2(\theta) = 1 - \sin(\theta)$. Then,

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{\pi} \Big[\big[r_1(\theta)\big]^2 - \big[r_2(\theta)\big]^2 \Big]\, d\theta, \\
&= \frac{1}{2}\int_0^{\pi} \Big[1 - \big[1 - \sin(\theta)\big]^2 \Big]\, d\theta, \\
&= \frac{1}{2}\int_0^{\pi} \big[1 - 1 - \sin^2(\theta) + 2\sin(\theta) \big]\, d\theta, \\
&= \frac{1}{2}\int_0^{\pi} \Big[-\frac{1}{2}\big(1 - \cos(2\theta)\big) + 2\sin(\theta) \Big]\, d\theta, \\
&= \frac{1}{4}\int_0^{\pi} \big[-1 + \cos(2\theta) + 4\sin(\theta) \big]\, d\theta, \\
&= \frac{1}{4}\Big[-\pi + \frac{1}{2}\sin(2\theta)\Big|_0^{\pi} - 4\cos(\theta)\Big|_0^{\pi} \Big], \\
&= \frac{1}{4}\big[-\pi - 4(-1 - 1) \big], \\
&= 2 - \frac{\pi}{4}.
\end{aligned}$$

$$\boxed{A = 2 - \frac{\pi}{4}}.$$

7. (8 points) Find the radius of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{n}{2}\left(\frac{x-1}{2}\right)^{n-1}$.

(8 points) Find the sum of this series for those values of x where the series converges.

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)}{2}\frac{|x-1|^n}{2^n}\frac{2}{n}\frac{2^{n-1}}{|x-1|^{n-1}} = \frac{|x-1|}{2}\frac{n+1}{n} = \frac{|x-1|}{2}\left(1+\frac{1}{n}\right)$$

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \frac{|x-1|}{2} < 1 \quad \Rightarrow \quad |x-1| < 2.$$

Therefore the radius of convergence is $R = 2$.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{n}{2}\left(\frac{x-1}{2}\right)^{n-1} &= \sum_{n=0}^{\infty}\left[\left(\frac{x-1}{2}\right)^{n}\right]', \\
&= \left[\sum_{n=0}^{\infty}\left(\frac{x-1}{2}\right)^{n}\right]', \\
&= \left[\frac{1}{1-\frac{(x-1)}{2}}\right]', \\
&= \left[\frac{2}{3-x}\right]', \\
&= \frac{2}{(3-x)^2}.
\end{aligned}$$

$$\boxed{\sum_{n=0}^{\infty} \frac{n}{2}\left(\frac{x-1}{2}\right)^{n-1} = \frac{2}{(3-x)^2}}.$$

8. (10 points) Find the Taylor polynomial of order 3 centered at $x = 1$ for the function $f(x) = \ln(x)$.

$$
\begin{aligned}
f(x) &= \ln(x), & f(1) &= 0, \\
f'(x) &= \frac{1}{x}, & f'(1) &= 1, \\
f''(x) &= -\frac{1}{x^2}, & f''(1) &= -1, \\
f'''(x) &= \frac{2}{x^3}, & f'''(1) &= 2,
\end{aligned}
$$

$$
T_3(x) = f(1) + f'(1)\,(x-1) + \frac{f''(1)}{2}\,(x-1)^2 + \frac{f'''(1)}{6}\,(x-1)^3,
$$

$$
\boxed{T_3(x) = (x-1) - \frac{1}{2}\,(x-1)^2 + \frac{1}{3}\,(x-1)^3}.
$$

9. (10 points)

(a) Find a solution $y(x)$ of the differential equation $(4+x^2)\,y' = xy$.

(b) Find the particular solution $y(x)$ of the above equation that satisfies $y(0) = 1$. Show all your work.

(a)

$$(4+x^2)\,y' = xy \quad \Rightarrow \quad \frac{y'}{y} = \frac{x}{4+x^2} \quad \Rightarrow \quad \int \frac{y'}{y}\,dx = \int \frac{x}{4+x^2}\,dx.$$

Substitute $u = y(x)$, then $du = y'\,dx$, and $v = 4+x^2$, then $dv = 2x\,dx$.

$$\int \frac{du}{u} = \frac{1}{2}\int \frac{dv}{v} \quad \Rightarrow \quad \ln(u) = \frac{1}{2}\ln(v) + c = \ln(\sqrt{v}) + c \quad \Rightarrow \quad \ln(y) = \ln\big[\sqrt{4+x^2}\big] + c,$$

$$\boxed{y(x) = \sqrt{4+x^2}\,e^c}.$$

(b)

$$1 = y(0) = \sqrt{4}\,e^c \quad \Rightarrow \quad e^c = \frac{1}{2} \quad \Rightarrow \quad \boxed{y(x) = \frac{1}{2}\sqrt{4+x^2}}.$$