Slide 1

Eigenvalues and Eigenvectors

- Review:
  - Formula for the inverse matrix.
  - Cramer's rule.
  - Determinants, areas and volumes.
- Definition of eigenvalues and eigenvectors.

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Review

Theorem 1 (Formula for the inverse matrix) *If A be an $n \times n$ matrix with $\det(A) = \Delta \neq 0$, then*

$$\left(A^{-1}\right)_{ij} = \frac{1}{\Delta}[C_{ji}].$$

where $C_{ij} = (-1)^{i+j}\det(A_{ij})$.

Theorem 2 (Cramer's rule) *If the matrix $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ is invertible, then the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every vector $\mathbf{b}$, given by*

$$x_i = \frac{1}{\Delta}\det(A_i(\mathbf{b})).$$

where x_i is the i component of $\mathbf{x}$, and $A_i(\mathbf{b}) = [\mathbf{a}_1, \cdots, \mathbf{b}, \cdots, \mathbf{a}_n]$, with $\mathbf{b}$ in the i column.

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Determinant, areas and volumes

Theorem 3 *Let* $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ *be an* $n \times n$ *matrix.*

If $n = 2$*, then* $|\det(A)|$ *is the area of the parallelogram determined by* $\mathbf{a}_1$*,* $\mathbf{a}_2$*.*

If $n = 3$*, then* $|\det(A)|$ *is the volume of the parallelepiped determined by* $\mathbf{a}_1$*,* $\mathbf{a}_2$*, and* $\mathbf{a}_3$*.*

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Determinant, areas and volumes

Sketch of the proof for $n = 2$. The elementary row operations

- add to one row a multiple of another row;
- switch two rows;

leave the absolute value of the determinant unchanged, and they also leave the area of the parallelogram unchanged.

These operations transform any parallelogram into a rectangle.

In the case of a rectangle, the determinant of the matrix constructed with the vectors that form the rectangle is the area of the rectangle.

Therefore, the theorem follows in the case $n = 2$.

Same argument holds for $n = 3$.

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Eigenvalues and eigenvectors

Definition 1 (Eigenvalues and eigenvectors) *Let A be an $n \times n$ matrix. A number λ is an eigenvalue of A if there exists a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ such that*

$$A\mathbf{x} = \lambda\mathbf{x}.$$

The vector $\mathbf{x}$ is called an eigenvalue of A corresponding to λ.

Notice: If $\mathbf{x}$ is an eigenvector, then $t\mathbf{x}$ with $t \neq 0$ is also an eigenvector.

Definition 2 (Eigenspace) *Let λ be an eigenvalue of A. The set of all vectors $\mathbf{x}$ solutions of $A\mathbf{x} = \lambda\mathbf{x}$ is called the eigenspace $E(\lambda)$.*

That is, $E(\lambda) = \{$ all eigenvectors with eigenvalue λ, and $\mathbf{0}\}$.

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Examples

- Consider the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

Show that the vectors $\mathbf{v}_1 = [1, 1]^T$ and $\mathbf{v}_2 = [1, -1]^T$ are eigenvectors of A and find the associated eigenvalues.

$$A\mathbf{v}_1 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4\mathbf{v}_1.$$

$$A\mathbf{v}_2 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -2\mathbf{v}_2.$$

Then, $\lambda_1 = 4$ and $\lambda_2 = -2$.

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Examples

- Is $\mathbf{v} = [1, 2]^T$ an eigenvector of matrix A given above?

The answer is no, because of the following calculation.

$$A\mathbf{v} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

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Examples

- Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}.$$

Show that the vectors $\mathbf{v}_1 = [-2, 1]^T$ is an eigenvector of A, and find the associated eigenvalue.

$$A\mathbf{v}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Therefore, $\lambda_1 = 0$.

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Examples

- Is there any other eigenvalue of the matrix A above?

One has to find the solutions of $A\mathbf{x} = \lambda\mathbf{x}$.

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 &= \lambda x_1 \\ 3x_1 + 6x_2 &= \lambda x_2 \quad \Rightarrow \quad x_1 + 2x_2 = \tfrac{\lambda}{3} x_2. \end{aligned}$$

Therefore $\lambda x_1 = \lambda x_2/3$, that is $\lambda\left(x_1 - \frac{1}{3}x_2\right) = 0$. This implies that $\lambda =$ or $3x_1 = x + 2$. The first case corresponds to the eigenvalue zero, already studied above, which has the eigenvector $\mathbf{v}_1 = [-2, 1]^T$.

The other case gives an eigenvector satisfying $3x_1 = x_2$, so one possible solution is

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

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Examples

We compute the eigenvalue associated to $\mathbf{v}_2 = [1, 3]^T$

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 21 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Therefore, the eigenvalue is $\lambda_2 = 7$.

This calculation seems complicated because one computes eigenvalues and eigenvectors at the same time. Later on we split the calculation, computing eigenvalues alone, and then eigenvectors.

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Eigenvalues and Eigenvectors

- Definition of eigenvalues and eigenvectors.
- Eigenspace.
- Geometrical interpretation of eigenvectors.
- Characteristic equation.

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Eigenvalues and eigenvectors

Definition 3 (Eigenvalues and eigenvectors) *Let A be an $n \times n$ matrix. A number λ is an eigenvalue of A if there exists a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ such that*

$$A\mathbf{x} = \lambda \mathbf{x}.$$

The vector $\mathbf{x}$ is called an eigenvalue of A corresponding to λ.

Notice: If $\mathbf{x}$ is an eigenvector, then $t\mathbf{x}$ with $t \neq 0$ is also an eigenvector.

Definition 4 (Eigenspace) *Let λ be an eigenvalue of A. The set of all vectors $\mathbf{x}$ solutions of $A\mathbf{x} = \lambda \mathbf{x}$ is called the eigenspace $E(\lambda)$.*

That is, $E(\lambda) = \{$ all eigenvectors with eigenvalue $\lambda,$ and $\mathbf{0}\}$.

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Examples of eigenspaces

- The matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ has eigenvectors $\lambda_1 = 4$ and $\lambda_2 = -2$. The corresponding eigenspaces are

$$E(4) = \{\mathbf{x} = t[1, 1]^T, \quad t \in I\!R\}.$$

$$E(-2) = \{\mathbf{x} = t[1, -1]^T, \quad t \in I\!R\}.$$

- The matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has eigenvectors $\lambda_1 = 0$ and $\lambda_2 = 7$. The corresponding eigenspaces are

$$E(0) = \{\mathbf{x} = t[-2, 1]^T, \quad t \in I\!R\}.$$

$$E(-2) = \{\mathbf{x} = t[1, 3]^T, \quad t \in I\!R\}.$$

Notice that not every eigenspace is one-dimensional.

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Eigenspace

Theorem 4 *Let λ be an eigenvector of A, an $n \times n$ matrix. Then, the set $E(\lambda) \subset I\!R^n$ is a subspace.*

Proof: Let $\mathbf{x}_1$, $\mathbf{x}_2 \in E(\lambda)$, that is,

$$A\mathbf{x}_1 = \lambda\mathbf{x}_1, \quad A\mathbf{x}_2 = \lambda\mathbf{x}_2.$$

Now compute

$$A(a\mathbf{x}_1 + b\mathbf{x}_2) = aA\mathbf{x}_1 + bA\mathbf{x}_2 = a\lambda\mathbf{x}_1 + b\lambda\mathbf{x}_2 = \lambda(a\mathbf{x}_1 + b\mathbf{x}_2).$$

Therefore, $a\mathbf{x}_1 + b\mathbf{x}_2 \in E(\lambda)$. $\square$

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Geometrical interpretation of eigenvectors

Think the $n \times n$ matrix A as a linear transformation $A : I\!R^n \to R^n$.

An eigenvector $\mathbf{x}$ of A determines a direction in $I\!R^n$ where the action of A is simple: It is a stretching or a compression, depending on whether $|\lambda| \geq 1$ or $|\lambda \leq 1$.

Theorem 5 *The eigenvalue of a diagonal $n \times n$ matrix are the elements of its diagonal, and its eigenvectors are the standard basis vectors $\mathbf{e}_i$, with $i = 1, \cdots, n$.*

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Theorem 6 *Let $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ be eigenvectors of A with eigenvalues $\{\lambda_1, \cdots, \lambda_r\}$, respectively.*

If the $\{\lambda_1, \cdots, \lambda_r\}$ are all different, then the $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ are l.i..

Proof: By induction in r.

For $r = 1$ the theorem is true.

Consider the case $r = 2$ as an intermediate step to understand the idea behind the proof. In this case we have two eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ corresponding to different eigenvalues λ_1, λ_2, that is $\lambda_1 \neq \lambda_2$. We have to show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ are l.i.. By contradiction, assume that they are l.d., that is, there exists a nonzero $a \in I\!R$ such that

$$\mathbf{v}_2 = a\mathbf{v}_1. \tag{1}$$

Apply the matrix A on both sides of the equation (1), then one gets $\lambda_2\mathbf{v}_2 = a\lambda_1\mathbf{v}_1$, because both vectors are eigenvectors of A. Now multiply equation (1) by λ_2. One gets, $\lambda_2\mathbf{v}_2 = a\lambda_2\mathbf{v}_1$.

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From these two equations one gets

$$0 = a(\lambda_2 - \lambda_1)\mathbf{v}_1.$$

Because $a \neq 0$, and the eigenvalues are different, one gets $\mathbf{v}_1 = 0$, which is a contradiction to the hypothesis that $\mathbf{v}_1$ is an eigenvector. Therefore, $\{\mathbf{v}_1, \mathbf{v}_2\}$ are l.i.. This is the idea of the proof, and we now repeat it in the case of r eigenvalues.

Assume that the theorem holds for $r-1$ eigenvectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_{r-1}\}$, and then show that it also holds for r eigenvectors vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$.

So assume that $\{\mathbf{v}_1, \cdots, \mathbf{v}_{r-1}\}$ are l.i., and that, by contradiction, suppose that the $\{\mathbf{v}_1, \cdots, \mathbf{v}_{r-1}, \mathbf{v}_r\}$ are l.d.. Then,

$$\mathbf{v}_r = a_1\mathbf{v}_1 + \cdots + a_{r-1}\mathbf{v}_{r-1},$$

with some $a_i \neq 0$.

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Apply the matrix A on both ides of equation above, then

$$\lambda_r\mathbf{v}_r = a_1\lambda_1\mathbf{v}_1 + \cdots + a_{r-1}\lambda_{r-1}\mathbf{v}_{r-1}.$$

Now multiply the first equation by λ_r,

$$\lambda_r\mathbf{v}_r = a_1\lambda_r\mathbf{v}_1 + \cdots + a_{r-1}\lambda_r\mathbf{v}_{r-1}.$$

Subtract these two equations, and then one gets

$$0 = a_1(\lambda_r - \lambda_1)\mathbf{v}_1 + \cdots + a_{r-1}(\lambda_r - \lambda_{r-1})\mathbf{v}_{r-1}.$$

Because all the λ_i are different, then the linear combination above says that the $\{\mathbf{v}_1, \cdots, \mathbf{v}_{r-1}\}$ are l.d.. But this contradicts the hypothesis.

Therefore, the $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ are l.i., and the theorem follows. $\square$

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The characteristic equation

To compute the eigenvalues and eigenvectors one has to solve the equation $A\mathbf{x} = \lambda\mathbf{x}$ for both, λ and $\mathbf{x}$. This equation is equivalent to

$$(A - \lambda I)\mathbf{x} = 0.$$

This homogeneous system has a non-zero solutions if and only if $\det(A - \lambda I) = 0$. Notice that this last equation is an equation only for the eigenvalues! There is no $\mathbf{x}$ in this equation, so one can solve only for λ and *then* solve for $\mathbf{x}$.

Definition 5 (Characteristic function) *Let A be an $n \times n$ matrix and I the $n \times n$ identity matrix. Then, the scalar function*

$$f(\lambda) = \det(A - \lambda I)$$

is called the characteristic function of A.

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The characteristic equation

Theorem 7 (Characteristic polynomial) *The characteristic function $f(\lambda)$ of an $n \times n$ matrix A is a polynomial in λ of degree n.*

Furthermore, the polynomial has the form

$$f(\lambda) = (-1)^n \lambda^n + \cdots + \det(A).$$

Example:

$$f(\lambda) = \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix}$$

$$\begin{aligned} f(\lambda) &= (a - \lambda)(d - \lambda) - bc, \\ &= \lambda^2 - (a + d)\lambda + (ad - bc). \end{aligned}$$

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The characteristic equation

Theorem 8 (Characteristic equation) *The number λ is an eigenvalue of A if and only if*

$$\det(A - \lambda I) = 0. \tag{2}$$

Equation (2) is called characteristic equation.

Proof:

$$A\mathbf{x} = \lambda\mathbf{x} \;\Leftrightarrow\; (A - \lambda I)\mathbf{x} = 0 \;\Leftrightarrow\; N(A - \lambda I) \neq \{\mathbf{0}\} \;\Leftrightarrow$$

$$(A - \lambda I) \text{ is not invertible } \;\Leftrightarrow\; \det(A - \lambda I) = 0.$$

□

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Examples

- Find the eigenvalues of $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ for any $a, b \in \mathbb{R}$ with $b \neq 0$.

Let start with matrix A,

$$0 = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 9, \;\Rightarrow \lambda = 1 \pm 3,$$

that is, $\lambda_1 = 4$, and $\lambda_2 = -2$. Now, in the case of matrix B one has

$$\begin{vmatrix} a-\lambda & b \\ -b & a-\lambda \end{vmatrix} = (a-\lambda)^2 + b^2 \neq 0,$$

therefore, B has none eigenvalues at all.

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Examples

Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$. The answer is:

$$0 = \begin{vmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2, \quad \Rightarrow \quad (\lambda - 2)^2 = 0,$$

that is, $\lambda = 2$. This eigenvalue has multiplicity 2, according to the following definition.

Definition 6 (Multiplicity of eigenvalues) *Let $f(\lambda)$ be the characteristic polynomial of an $n \times n$ matrix. The eigenvalue λ_0 has algebraic multiplicity $r > 0$ if and only if*

$$f(\lambda) = (\lambda - \lambda_0)^r g(\lambda), \quad \textit{with} \quad g(\lambda_0) \neq 0.$$

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Examples

- Find the eigenvalues and eigenspaces of the following two matrices:

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The both matrices have the same eigenvalues, because,

$$f_A(\lambda) = f_B(\lambda) = (\lambda - 3)^2 (1 - \lambda)$$

so the eigenvalues are:

- $\lambda = 3$ with multiplicity 2;
- $\lambda = 1$ with multiplicity 1.

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Examples

One can check that the eigenspaces are the following:

$$E_A(3) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad E_A(1) = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\},$$

$$E_B(3) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad E_B(1) = \left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \right\}.$$

Notice: $\dim E(\lambda) \leq \text{multipl.}(\lambda)$.

In the case of B, where $\dim E_B(\lambda) = \text{multipl.}(\lambda)$ for every eigenvalue of B, the set of all eigenvectors of B is a basis of $\mathbb{R}^3$.

In the case of A, where for $\lambda = 3$ holds that $\dim E_A(3) < \text{multipl.}(3)$, the set of eigenvectors of A is not a basis of $\mathbb{R}^3$.

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Diagonalization

- Diagonalization and eigenvectors.
- Application: Computing powers of a matrix.
- Examples.

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Diagonalizable

Definition 7 (Diagonalizable matrices) *An $n \times n$ matrix A is diagonalizable if there exists a diagonal matrix D and an invertible matrix P, with inverse P^{-1}, such that*

$$A = PDP^{-1}. \tag{3}$$

Notice that D is a diagonal matrix if

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = \text{diag}[d_1, \cdots, d_n].$$

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Diagonalization and eigenvectors

Notice that if $B = [\mathbf{b}_1, \cdots, \mathbf{b}_n]$, then $BD = [d_1\mathbf{b}_1, \cdots, d_n\mathbf{b}_n]$. Also notice that for a general B holds $BD \neq DB$.

Finally recall that $AB = [A\mathbf{b}_1, \cdots, A\mathbf{b}_n]$.

Now, the main result:

Theorem 9 (Diagonalization and eigenvectors) *An $n \times n$ matrix A is diagonalizable if and only if A has n l.i. eigenvectors.*

Furthermore, if we write $A = PDP^{-1}$, with D diagonal, then $P = [\mathbf{p}_1, \cdots, \mathbf{p}_n]$, and $D = diag[\lambda_1, \cdots, \lambda_n]$, where

$$A\mathbf{p}_i = \lambda_i \mathbf{p}_i, \quad i = 1, \cdots, n,$$

that is, $\{\mathbf{p}_1, \cdots, \mathbf{p}_n\}$ are the eigenvectors with eigenvectors $\lambda_1, \cdots, \lambda_n$, respectively.

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Proof:

($\Leftarrow$) Let $P = [\mathbf{p}_1, \cdots, \mathbf{p}_n]$ be a matrix formed with the n eigenvectors of A, and denote by λ_i the corresponding eigenvalues, that is, $A\mathbf{p}_i = \lambda_i \mathbf{p}_i$. Because $\{\mathbf{p}_1, \cdots, \mathbf{p}_n\}$ are l.i. then P is invertible. Introduce the diagonal matrix $D = \text{diag}[\lambda_1, \cdots, \lambda_n]$. Then,

$$PD = [\lambda_1 \mathbf{p}_1, \cdots, \lambda_n \mathbf{p}_n] = [A\mathbf{p}_1, \cdots, A\mathbf{p}_n] = AP.$$

Therefore, $A = PDP^{-1}$.

($\Rightarrow$) Given the invertible matrix P, introduce its column vectors $P = [\mathbf{p}_1, \cdots, \mathbf{p}_n]$. Denote the diagonal matrix $D = \text{diag}[d_1, \cdots, d_n]$. Now, the equation $A = PDP^{-1}$ implies $AP = AD$, that is,

$$[A\mathbf{p}_1, \cdots, A\mathbf{p}_n] = [d_1 \mathbf{p}_1, \cdots, d_n \mathbf{p}_n],$$

which says that $A\mathbf{p}_i = d_i \mathbf{p}_i$ for every $i = 1, \cdots, n$. So the $\mathbf{p}_i$ are eigenvectors of A with eigenvalue d_i. And these vectors are l.i. because P is invertible. $\square$

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Example

Recall the matrix A given by

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix},$$

which has eigenvectors and eigenvalues given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda_1 = 4, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = -2.$$

Then,

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad P^{-1} = -\frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}.$$

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Example

Then, it is easy to check that

$$\begin{aligned} PDP^{-1} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \left(-\frac{1}{2}\right) \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}, \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}, \\ &= \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \\ &= A. \end{aligned}$$

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Applications

Theorem 10 (Powers of matrices) *Let A be an $n \times n$ matrix. If A is diagonalizable, then*

$$A^k = P(D^k)P^{-1}.$$

Proof:

$$\begin{aligned} A^k &= (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1}), \\ &= PD(P^{-1}P)D(P^{-1}P) \cdots (P^{-1}P)DP^{-1}, \\ &= P(D^k)P^{-1}. \end{aligned}$$

□

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Example

- Compute A^4 where

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

We know that the eigenvectors and eigenvalues of A are given by

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad \lambda_1 = 0, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \lambda_2 = 7.$$

Then,

$$P = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix}, \quad P^{-1} = -\frac{1}{7}\begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}.$$

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Example

$$\begin{aligned} A^4 &= \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 7^4 \end{bmatrix}\left(-\frac{1}{7}\right)\begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix}, \\ &= \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 7^3 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix}, \\ &= \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 7^3 & 2(7^3) \end{bmatrix}, \\ &= \begin{bmatrix} 7^3 & 2(7^3) \\ 3(7^3) & 6(7^3) \end{bmatrix}, \\ &= 7^3\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \\ &= 7^3 A. \end{aligned}$$

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Inner product

- Definition of inner product.
- Examples.
- Norm, distance.
- Orthogonal vectors.
- Orthogonal complement.

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Definition of inner product

Definition 8 (Inner product) *Let V be a vector space over $\mathbb{R}$. An inner product $(\, , \,)$ is a function $V \times V \to \mathbb{R}$ with the following properties*

1. $\forall \, \mathbf{u} \in V$, $(\mathbf{u}, \mathbf{u}) \geq 0$, *and* $(\mathbf{u}, \mathbf{u}) = 0 \Leftrightarrow \mathbf{u} = \mathbf{0}$;
2. $\forall \, \mathbf{u}$, $\mathbf{v} \in V$, *holds* $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$;
3. $\forall \, \mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$, *and* $\forall \, a$, $b \in \mathbb{R}$ *holds* $(a\mathbf{u} + b\mathbf{v}, \mathbf{w}) = a(\mathbf{u}, \mathbf{w}) + b(\mathbf{v}, \mathbf{w})$.

Notation: V together with $(\, , \,)$ is called an inner product space.

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Examples

- Let $V = I\!R^n$, and $\{\mathbf{e}_i\}_{i=1}^n$ be the standard basis. Given two arbitrary vectors $\mathbf{x} = \sum_{i=1}^n x_i\mathbf{e}_i$ and $\mathbf{y} = \sum_{i=1}^n y_i\mathbf{e}_i$, then

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i y_i.$$

  This product is also denoted as $\sum_{i=1}^n x_i y_i = \mathbf{x} \cdot \mathbf{y}$. It is called Euclidean inner product.

- Let $V = I\!R^2$, and $\{\mathbf{e}_i\}_{i=1}^2$ be the standard basis. Given two arbitrary vectors $\mathbf{x} = \sum_{i=1}^2 x_i\mathbf{e}_i$ and $\mathbf{y} = \sum_{i=1}^2 y_i\mathbf{e}_i$, then

$$(\mathbf{x}, \mathbf{y}) = 2x_1y_1 + 3x_2y_2.$$

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Examples

- Let $V = C([0, 1])$. Given two arbitrary vectors $f(x)$ and $g(x)$, then

$$(f, g) = \int_0^1 f(x)g(x)\,dx.$$

- Let $V = C([0, 1])$. Given two arbitrary vectors $f(x)$ and $g(x)$, then

$$(f, g) = \int_0^1 e^x f(x)g(x)\,dx.$$

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Norm

An inner product space induces a norm, that is, a notion of length of a vector.

Definition 9 (Norm) *Let V, $(\, , \,)$ be a inner product space. The norm function, or length, is a function $V \to \mathbb{R}$ denoted as $\| \; \|$, and defined as*

$$\|\mathbf{u}\| = \sqrt{(\mathbf{u}, \mathbf{u})}.$$

Example: The Euclidean norm is

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{(x_1)^2 + \cdots + (x_n)^2}.$$

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Distance

A norm in a vector space, in turns, induces a notion of distance between two vectors, defined as the length of their difference..

Definition 10 (Distance) *Let V, $(\, , \,)$ be a inner product space, and $\| \; \|$ be its associated norm. The distance between $\mathbf{u}$ and $\mathbf{v} \in V$ is given by*

$$dist(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

Example: The Euclidean distance between to points $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^3$ is

$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

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Angle between vectors

Definition 11 (Angle) *Let V, $(\ ,\ )$ be a inner product space, and $\|\ \|$ be its associated norm. The angle θ between two vectors $\mathbf{u}$, $\mathbf{v} \in V$ is defined as*

$$\cos(\theta) = \frac{(\mathbf{u}, \mathbf{v})}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

Examples:

- The angle θ between two vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^2$ with respect to the Euclidean inner product is given by

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\| \cos(\theta).$$

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Angle between vectors

- The notion of angle can also be introduced in the inner space of continuous functions $C([0,1])$. The angle between $f(x) = x$ and $g(x) = x^2$ is approximately 14.5, because

$$(f, g) = \int_0^1 x^3 dx = \frac{1}{4},$$

$$\|f\| = \sqrt{\int_0^1 x^2 dx} = \frac{1}{\sqrt{3}}, \quad \|g\| = \sqrt{\int_0^1 x^4 dx} = \frac{1}{\sqrt{5}}.$$

then,

$$\cos(\theta) = \frac{(f, g)}{\|f\|\,\|g\|} = \frac{\sqrt{15}}{4} \quad \Leftarrow \quad \theta \simeq 14.5.$$

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Orthogonal vectors

Definition 12 (Orthogonal vectors) *Let V, $(\ ,\ )$ be an inner product space. Two vectors $\mathbf{u}$, $\mathbf{v} \in V$ are orthogonal, or perpendicular, if and only if*

$$(\mathbf{u}, \mathbf{v}) = 0.$$

We call them orthogonal, because $\cos(\theta) = 0$, which implies $\theta = \pi/2$. Example:

- The vectors $\cos(x)$, $\sin(x) \in C([0, 2\pi])$ are orthogonal, because

$$\begin{aligned}(\cos(x), \sin(x)) &= \int_0^{2\pi} \sin(x)\cos(x)\,dx, \\ &= \frac{1}{2}\int_0^{2\pi} \sin(2x)\,dx, \\ &= -\frac{1}{4}\left(\cos(2x)|_0^{2\pi}\right), \\ &= 0.\end{aligned}$$