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Determinants (Sec. 3.2)

- Review: Definition of determinant of $n \times n$ matrices.
- Properties of determinants.
- Determinants and elementary row operations.
- Determinant of a product of matrices.

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Review: Definition of determinant

Definition 1 *The determinant of an $n \times n$ matrix $A = [a_{ij}$ is given by*

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} \det(A_{1j}) a_{1j}.$$

This formula is called "expansion by the first row."

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Properties

Theorem 1 (Main properties of $n \times n$ determinants) *Let $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ be an $n \times n$ matrix. Let $\mathbf{c}$ be an n-vector.*

- $\det([\mathbf{a}_1, \cdots, \mathbf{a}_j + \mathbf{c}, \cdots, \mathbf{a}_n]) = \det([\mathbf{a}_1, \cdots, \mathbf{a}_j \cdots, \mathbf{a}_n]) + \det([\mathbf{a}_1, \cdots, \mathbf{c}, \cdots, \mathbf{a}_n])$.
- $\det([\mathbf{a}_1, \cdots, c\mathbf{a}_j, \cdots, \mathbf{a}_n]) = c \det([\mathbf{a}_1, \cdots, \mathbf{a}_j, \cdots, \mathbf{a}_n])$.
- $\det([\mathbf{a}_1, \cdots, \mathbf{a}_i, \cdots, \mathbf{a}_j, \cdots, \mathbf{a}_n]) = -\det([\mathbf{a}_1, \cdots, \mathbf{a}_j, \cdots, \mathbf{a}_i, \cdots, \mathbf{a}_n])$.
- $\det([\mathbf{a}_1, \cdots, \mathbf{a}_i, \cdots, \mathbf{a}_i, \cdots, \mathbf{a}_n]) = 0$.
- $\det(A) = \det(A^T)$.
- $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ *are l.d.* $\Leftrightarrow \det([\mathbf{a}_1, \cdots, \mathbf{a}_n]) = 0$.
- A *is invertible* $\Leftrightarrow \det(A) \neq 0$.

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Properties

The properties of the determinant on the column vectors of A and the property $\det(A) = \det(A^T)$ imply the following results on the rows of A.

Theorem 2 (Determinants and elementary row operations) *Let A be a $n \times n$ matrix.*

- *Let B be the result of adding to a row in A a multiple of another row in A. Then,* $\det(B) = \det(A)$.
- *Let B be the result of interchanging two rows in A. Then,* $\det(B) = -\det(A)$.
- *Let B be the result of multiply a row in A by a number k. Then,* $\det(B) = k \det(A)$.

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Determinant and elementary row operations

Theorem 3 *If E represents an elementary row operation and A is an $n \times n$ matrix, then*

$$\det(EA) = \det(E)\det(A).$$

The proof is to compute the determinant of every elementary row operation matrix, E, and then use the previous theorem.

Theorem 4 (Determinant of a product) *If A, B are arbitrary $n \times n$ matrices, then*

$$\det(AB) = \det(A)\det(B).$$

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Determinant of a product of matrices

Proof: If A is not invertible, then AB is not invertible, then the theorem holds, because $0 = \det(AB) = \det(A)\det(B) = 0$. Suppose that A is invertible. Then there exist elementary row operations $E_k, \cdots, E_1$ such that

$$A = E_k \cdots E_1.$$

Then,

$$\begin{aligned} \det(AB) &= \det(E_k \cdots E_1 B), \\ &= \det(E_k)\det(E_{k-1} \cdots E_1 B), \\ &= \det(E_k) \cdots \det(E_1)\det(B), \\ &= \det(E_k \cdots E_1)\det(B), \\ &= \det(A)\det(B). \end{aligned}$$

□

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Formula for the inverse matrix

- Formula for the inverse matrix.
- Application to systems of linear equations.

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Formula for the inverse matrix

Theorem 5 *Let A be an $n \times n$ matrix with components $(A)_{ij} = a_{ij}$. Let $C_{ij} = (-1)^{i+j}\det(A_{ij})$ be the ijth cofactor, and $\Delta = \det(A)$. Then the component ij of the inverse matrix A^{-1} is given by*

$$\left(A^{-1}\right)_{ij} = \frac{1}{\Delta}[C_{ji}].$$

That is,

$$A^{-1} = \frac{1}{\Delta}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

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Formula for the inverse matrix

Proof: It is a straightforward computation. Let us denote B the matrix with components $(B)_{ij} = C_{ji}/\Delta$. Then,

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \frac{1}{\Delta} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Compute each component of the product AB.

$$(AB)_{11} = \frac{1}{\Delta}(C_{11}a_{11} + C_{12}a_{12} + \cdots + C_{1n}a_{1n}) = 1,$$

because the factor in the numerator in the right hand side is precisely $\det(A) = \Delta$.

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The second component is given by

$$(AB)_{12} = \frac{1}{\Delta}(C_{11}a_{21} + C_{12}a_{22} + \cdots + C_{1n}a_{2n}).$$

The factor between brackets in the right hand side is an expansion by the first row of the determinant of a matrix whose first row is

$$a_{21}, a_{22}, \cdots a_{2n}.$$

That is,

$$(AB)_{12} = \frac{1}{\Delta} \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0.$$

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An analogous calculation shows that $(AB)_{ij}$ is given by

$$(AB)_{ij} = \frac{1}{\Delta}(C_{j1}a_{i1} + C_{j2}a_{i2} + \cdots + C_{jn}a_{in}),$$

The factor between brackets in the right hand side is an expansion by the j row of the determinant of a matrix whose j row is is the i row of A,

$$a_{i1}, a_{i2}, \cdots a_{in}.$$

That is,

$$(AB)_{ij} = \frac{1}{\Delta}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \text{in the } j\text{-row}$$

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Therefore, when $i \neq j$ the factor between brackets is the determinant of a matrix with two identical rows, so $(AB)_{ij} = 0$ for $i \neq j$. If $i = j$, the the that factor is precisely $\det(A)$, then $(AB)_{ii} = 1$.

Summarizing,

$$\begin{aligned} (AB)_{ij} &= \frac{1}{\Delta}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \text{in the } j\text{-row} \\ &= I_{ij} \end{aligned}$$

Repeat this calculation for BA. □

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Systems of linear equations

Theorem 6 *Suppose that the matrix A is invertible. Then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution for every vector $\mathbf{b}$. If x_i are the components of $\mathbf{x}$ and $A_i(\mathbf{b}) = [\mathbf{a}_1, \cdots, \mathbf{b}, \cdots, \mathbf{a}_n]$, with $\mathbf{b}$ in the i column, then*

$$x_i = \frac{1}{\Delta} \det(A_i(\mathbf{b})).$$

Proof: A invertible means that the solution can be written as $\mathbf{x} = A^{-1}\mathbf{b}$. From the formula of the inverse matrix one has that

$$x_i = \frac{1}{\Delta}(C_{1i}b_1 + C_{2i}b_2 + \cdots + C_{ni}b_n),$$

where b_i are the components of $\mathbf{b}$. Notice that if one expands the $\det(A_i(\mathbf{b}))$ by the i row one gets

$$\det(A_i(\mathbf{b})) = (C_{1i}b_1 + C_{2i}b_2 + \cdots + C_{ni}b_n).$$

□