

Review: Isomorphism

**Definition 1 (Isomorphism)** The linear transformation  $T: V \to W$  is an isomorphism if T is one-to-one and onto.

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Example:  $T: P_1 \to \mathbb{I}\!\!R^3$  given by

$$T(a+bt+ct^2) = \begin{vmatrix} a \\ b \\ c \end{vmatrix}$$

is an isomorphism.

**Definition 2 (Dimension)** A vector space V has dimension n if the maximum number of l.i. vectors is n.

Basis and components

**Definition 3 (Basis)** A basis of an n-dimensional vector space V is any set  $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  of n l.i. vectors.

**Theorem 1 (Basis)** Let V be an n-dimensional vector space. The set  $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  is a basis of  $V \Leftrightarrow \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  are l.i. and they span V.

**Theorem 2** Let  $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  be a basis of V. Then, each vector  $\mathbf{v} \in V$  has a unique decomposition

$$\mathbf{v} = a_1 \mathbf{u}_1 + \dots + a_n \mathbf{v}_n.$$

Proof of Theorem 1:

 $(\Rightarrow)$  Suppose that  $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  does not span V. Then there exists  $\mathbf{v} \in V$  that it is not a linear combination of  $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ . That is,  $b\mathbf{v} + a_1\mathbf{u}_1 + \dots + a_n\mathbf{u}_n = 0$  implies that  $b = a_1 = \dots = a_n = 0$ . This in turn says that  $\{\mathbf{v}, \mathbf{u}_1, \dots, \mathbf{u}_n\}$  is a l.i. set. But this is a contradiction with the assumption that n is the maximum number of l.i. vectors in V.

 $(\Leftarrow)$  { $\mathbf{u}_1, \dots, \mathbf{u}_n$ } is l.i. and spans V. Then, for all  $\mathbf{v} \in V$  there exists numbers  $a_1, \dots, a_n$  such that  $\mathbf{v} = a_1 \mathbf{u}_1 + \dots + a_n \mathbf{u}_n$ . That is, the set { $\mathbf{v}, \mathbf{u}_1, \dots, \mathbf{u}_n$ } is l.d. for all  $\mathbf{v} \in V$ . That says that n is the maximum number of l.i. vectors in V.

Proof of Theorem 2: The set  $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  is a basis of V so they span V. Then, there exists scalars  $a_i$ , for  $1 \leq i \leq n$  such that the following decomposition holds,

$$\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{u}_i.$$

This decomposition is unique. Because, if there is another decomposition

$$\mathbf{v} = \sum_{i=1}^{n} b_i \mathbf{u}_i.$$

then the difference has the form

$$\sum_{i=1}^{n} (a_i - b_i) \mathbf{u}_i = 0.$$

Because the vectors  $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$  are l.i. this implies that  $a_i = b_i$  for all  $0 \le i \le n$ .

*Exercises* Consider the basis  $\{\mathbf{e}_1, \mathbf{e}_2\}$  given by  $\mathbf{e}_1 = \begin{bmatrix} 1\\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0\\ 1 \end{bmatrix}.$ 

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Consider a second basis  $\{\mathbf{u}_1,\mathbf{u}_2\}$  given by

$$\mathbf{u}_1 = \begin{bmatrix} 1\\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1\\ -1 \end{bmatrix}$$

Find the components of  $\mathbf{x} = \mathbf{e}_1 + 2\mathbf{e}_2$  in the basis  $\{\mathbf{u}_1, \mathbf{u}_2\}$ .

$$\mathbf{x} = \mathbf{e}_1 + 2\mathbf{e}_2 = \begin{bmatrix} 1\\2 \end{bmatrix}_e, \quad [\mathbf{x}]_e = \begin{bmatrix} 1\\2 \end{bmatrix}_e$$

The vectors  $\{\mathbf{u}_1, \mathbf{u}_2\}$  form a basis so there exists constants  $c_1, c_2$  such that

$$\mathbf{x} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_u, \quad [\mathbf{x}]_u = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_e$$

Therefore,

$$[\mathbf{x}]_e = c_1[\mathbf{u}_1]_e + c_2[\mathbf{u}_2]_e.$$

That is,

$$\begin{bmatrix} 1\\2 \end{bmatrix}_e = \begin{bmatrix} 1 & 1\\1 & -1 \end{bmatrix} \begin{bmatrix} c_1\\c_2 \end{bmatrix}_u$$

Then one has to solve the augmented matrix

$$\begin{bmatrix} 1 & 1 & | & 1 \\ 1 & -1 & | & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & | & \frac{3}{2} \\ 0 & 1 & | & -\frac{1}{2} \end{bmatrix},$$

so  $c_1 = 3/2$  and  $c_2 = -1/2$ , and then

$$[\mathbf{x}]_y = \begin{bmatrix} 1\\2 \end{bmatrix}_e, \quad [\mathbf{x}]_u = \begin{bmatrix} \frac{3}{2}\\-\frac{1}{2} \end{bmatrix}_u.$$

Change of basis

Theorem 3 (Change of basis) Let  $\{\mathbf{u}_1,\cdots,\mathbf{u}_n\}$  and  $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$  be basis of V. Then, there exists a unique  $n \times n$ invertible matrix  $P_{v \leftarrow u}$  such that

 $[\mathbf{x}]_v = P_{v \leftarrow u}[\mathbf{x}]_u,$ for all  $\mathbf{x} \in V$ . Furthermore, the matrix  $P_{v \leftarrow u}$  has the form  $P_{v \leftarrow u} = [[\mathbf{u}_1]_v, \cdots, [\mathbf{u}_n]_v].$ 

$$P_{v \leftarrow u} = [[\mathbf{u}_1]_v, \cdots, [\mathbf{u}_n]_v]$$

*Proof of Theorem 3:* Both sets  $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$  and  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  are basis of V, then there exist a unique set of numbers  $\{u_1, \cdots, u_n\}$  and  $\{v_1, \cdots, v_n\}$  such that

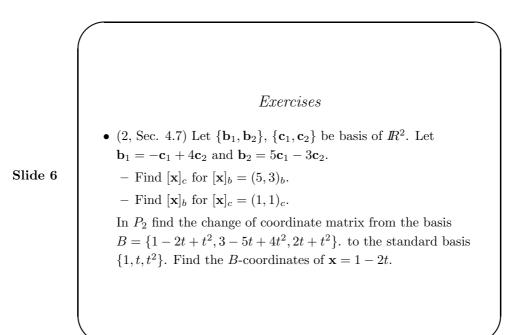
$$\mathbf{x} = u_1 \mathbf{u}_1 + \dots + u_n \mathbf{u}_n = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}_u, \quad \mathbf{x} = v_1 \mathbf{v}_1 + \dots + v_n \mathbf{v}_n = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}_v.$$

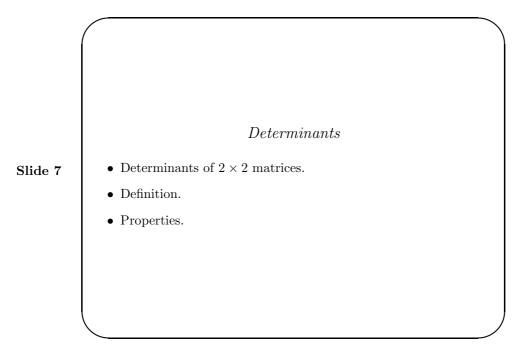
Therefore,

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$$\begin{bmatrix} v_1\\ \vdots\\ v_n \end{bmatrix}_v = [[\mathbf{u}_1]_v, \cdots, [\mathbf{u}_n]_v] \begin{bmatrix} u_1\\ \vdots\\ u_n \end{bmatrix}_u.$$

This system of equations for  $(u_1, \cdots u_n)$  has a unique solution solutions for all  $(v_1, \cdots v_n)$ , because the **u**'s and **v**'s are basis. That is,  $P_{v\leftarrow u} = [[\mathbf{u}_1]_v, \cdots, [\mathbf{u}_n]_v]$  is invertible. 

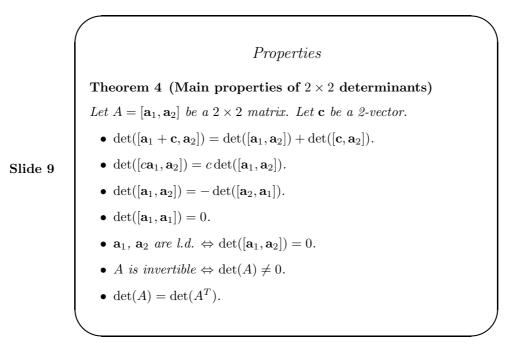


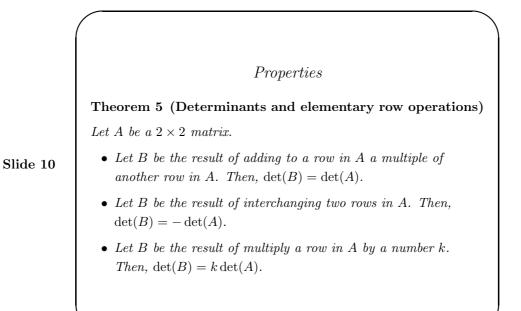


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 $2 \times 2 \ determinant$  **Definition 4** The determinant of  $a \ 2 \times 2 \ matrix \ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is given by  $\Delta = \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$ 

The determinant appears in the computation of the inverse matrix.





## Determinants and areas

**Theorem 6** Let  $A = [\mathbf{a}_1, \mathbf{a}_2]$  be a 2 × 2 matrix, with  $\mathbf{a}_1$  and  $\mathbf{a}_2$  being nonzero and noncollinear. Then,  $|\det([\mathbf{a}_1, \mathbf{a}_2])|$  is the area of the parallelogram formed by  $\mathbf{a}_1$  and  $\mathbf{a}_2$ .

*Proof:* Choose a basis  $\mathbf{e}_1$ ,  $\mathbf{e}_2$  such that  $\mathbf{a}_1 = b\mathbf{e}_1$ , for some number  $b \neq 0$ . Because  $\mathbf{a}_1$  is not collinear to  $\mathbf{a}_2$ , there exists a  $c \neq 0$  such that the vector  $\mathbf{u} = c\mathbf{a}_1 + \mathbf{a}_2$  is collinear to  $\mathbf{e}_2$ . For that vector  $\mathbf{u}$  holds that  $\mathbf{u} = h\mathbf{e}_2$ , where |h| is the height of the parallelogram. Summarizing:

$$\mathbf{a}_1 = \begin{bmatrix} b \\ 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 0 \\ h \end{bmatrix}$$

Now, the determinant of A is the following:

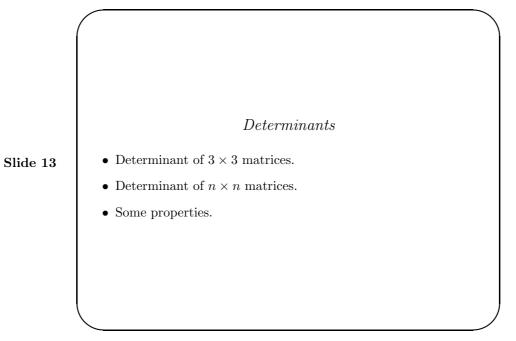
 $\det(A) = \det([\mathbf{a}_1, \mathbf{a}_2]) = \det([\mathbf{a}_1, \mathbf{a}_2 + c\mathbf{a}_1]) = \det([\mathbf{a}_1, \mathbf{u}]).$ 

Therefore,

$$\det(A) = \begin{vmatrix} b & 0 \\ 0 & h \end{vmatrix} = hb.$$

Then,  $|\det(A)| = |h| |b|$ , where |h| is the height and |b| the length of the base of the parallelogram.

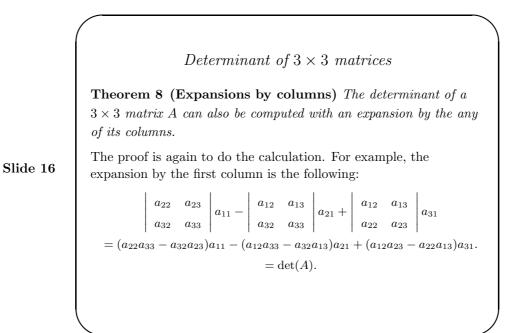
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Slide 14  $Determinant of 3 \times 3 matrices$   $Definition 5 The determinant of a 3 \times 3 matrix A is given by$   $det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$   $= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} a_{11} - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} a_{12} + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} a_{13},$   $= (a_{22}a_{33} - a_{32}a_{23})a_{11} - (a_{21}a_{33} - a_{31}a_{23})a_{12} + (a_{21}a_{32} - a_{31}a_{22})a_{13}.$ This formula is called "expansion by the first row."

> Determinant of  $3 \times 3$  matrices **Theorem 7 (Expansions by rows)** The determinant of  $a 3 \times 3$ matrix A can also be computed with an expansion by the second row or by the third row. The proof is just do the calculation. For example, the expansion by the second row is the following:  $-\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} a_{21} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} a_{22} - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} a_{23}$   $= -(a_{12}a_{33} - a_{32}a_{13})a_{21} + (a_{11}a_{33} - a_{31}a_{13})a_{22} - (a_{11}a_{32} - a_{31}a_{12})a_{23}.$  $= \det(A).$



Determinant of  $n \times n$  matrices Notation:  $A_{ij} = \begin{bmatrix} a_{11} & \cdots & \mathbf{a_{1j}} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ \mathbf{a_{i1}} & \cdots & \mathbf{a_{ij}} & \cdots & \mathbf{a_{in}} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & \mathbf{a_{nj}} & \cdots & a_{nn} \end{bmatrix}$  $A = \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \end{bmatrix}, \text{ Sign of coefficient } a_{ij} \text{ is } (-1)^{i+j}.$ 

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**Definition 6** The determinant of an  $n \times n$  matrix  $A = [a_{ij} \ is$ given by  $\det(A) = \det(A_{11})a_{11} - \det(A_{12})a_{12} + \dots + (-1)^{1+n} \det(A_{1n})a_{1n},$  $= \sum_{j=1}^{n} (-1)^{1+j} \det(A_{1j})a_{1j}.$ 

Determinant of  $n \times n$  matrices

This formula is called "expansion by the first row."

Determinant of  $n \times n$  matrices

**Theorem 9** The determinant of an  $n \times n$  matrix  $A = [a_{ij}]$  can be computed by an expansion along any row or along any column. That is,

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$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \det(A_{ij}) a_{ij}, \text{ for any } i = 1, \dots, n,$$
$$= \sum_{i=1}^{n} (-1)^{i+j} \det(A_{ij}) a_{ij}, \text{ for any } j = 1, \dots, n.$$

Notation: The cofactor  $C_{ij}$  of a matrix A is the number given by

$$C_{ij} = (-1)^{i+j} \det(A_{ij}).$$

**Theorem 10**  $det(A) = det(A^T)$ .

## Determinant of $n \times n$ matrices

Suggestion: If a matrix has a row or a column with several zeros, then it is simpler to compute its determinant by an expansion along that row or column.

**Theorem 11** The determinant of a triangular matrix is the product of its diagonal elements.

Examples:  

$$\begin{vmatrix}
1 & 2 & 3 \\
0 & 4 & 5 \\
0 & 0 & 6
\end{vmatrix} = \begin{vmatrix}
4 & 5 \\
0 & 6
\end{vmatrix} (1) = 1 \times 4 \times 6 = 24.$$

$$\begin{vmatrix}
1 & 0 & 0 \\
2 & 3 & 0 \\
4 & 1 & 5
\end{vmatrix} = \begin{vmatrix}
3 & 0 \\
1 & 5
\end{vmatrix} (1) = 1 \times 3 \times 5 = 15.$$