Slide 1

Components and change of basis

- Review: Isomorphism.
- Review: Components in a basis.
- Unique representation in a basis.
- Change of basis.

Slide 2

Review: Isomorphism

Definition 1 (Isomorphism) *The linear transformation $T : V \to W$ is an isomorphism if T is one-to-one and onto.*

Example: $T : P_1 \to \mathbb{R}^3$ given by

$$T(a + bt + ct^2) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

is an isomorphism.

Slide 3

Basis and components

Definition 2 (Dimension) *A vector space V has dimension n if the maximum number of l.i. vectors is n.*

Definition 3 (Basis) *A basis of an n-dimensional vector space V is any set $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ of n l.i. vectors.*

Theorem 1 (Basis) *Let V be an n-dimensional vector space. The set $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ is a basis of V $\Leftrightarrow$ $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ are l.i. and they span V.*

Theorem 2 *Let $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be a basis of V. Then, each vector $\mathbf{v} \in V$ has a unique decomposition*

$$\mathbf{v} = a_1\mathbf{u}_1 + \cdots + a_n\mathbf{v}_n.$$

Proof of Theorem 1:

($\Rightarrow$) Suppose that $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ does not span V. Then there exists $\mathbf{v} \in V$ that it is not a linear combination of $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$. That is, $b\mathbf{v} + a_1\mathbf{u}_1 + \cdots + a_n\mathbf{u}_n = 0$ implies that $b = a_1 = \cdots = a_n = 0$. This in turn says that $\{\mathbf{v}, \mathbf{u}_1, \cdots, \mathbf{u}_n\}$ is a l.i. set. But this is a contradiction with the assumption that n is the maximum number of l.i. vectors in V.

($\Leftarrow$) $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ is l.i. and spans V. Then, for all $\mathbf{v} \in V$ there exists numbers $a_1, \cdots a_n$ such that $\mathbf{v} = a_1\mathbf{u}_1 + \cdots + a_n\mathbf{u}_n$. That is, the set $\{\mathbf{v}, \mathbf{u}_1, \cdots, \mathbf{u}_n\}$ is l.d. for all $\mathbf{v} \in V$. That says that n is the maximum number of l.i. vectors in V. $\square$

Proof of Theorem 2: The set $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ is a basis of V so they span V. Then, there exists scalars a_i, for $1 \leq i \leq n$ such that the following decomposition holds,

$$\mathbf{v} = \sum_{i=1}^{n} a_i\mathbf{u}_i.$$

This decomposition is unique. Because, if there is another decomposition

$$\mathbf{v} = \sum_{i=1}^{n} b_i\mathbf{u}_i.$$

then the difference has the form

$$\sum_{i=1}^{n} (a_i - b_i)\mathbf{u}_i = 0.$$

Because the vectors $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ are l.i. this implies that $a_i = b_i$ for all $0 \leq i \leq n$. $\square$

Slide 4

Exercises

Consider the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ given by

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Consider a second basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ given by

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find the components of $\mathbf{x} = \mathbf{e}_1 + 2\mathbf{e}_2$ in the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$.

$$\mathbf{x} = \mathbf{e}_1 + 2\mathbf{e}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_e, \quad [\mathbf{x}]_e = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_e.$$

The vectors $\{\mathbf{u}_1, \mathbf{u}_2\}$ form a basis so there exists constants c_1, c_2 such that

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_u, \quad [\mathbf{x}]_u = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_e.$$

Therefore,

$$[\mathbf{x}]_e = c_1[\mathbf{u}_1]_e + c_2[\mathbf{u}_2]_e.$$

That is,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}_e = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_u$$

Then one has to solve the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \end{array}\right],$$

so $c_1 = 3/2$ and $c_2 = -1/2$, and then

$$[\mathbf{x}]_y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_e, \quad [\mathbf{x}]_u = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}_u.$$

Slide 5

Change of basis

Theorem 3 (Change of basis) *Let* $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ *and* $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ *be basis of* V*. Then, there exists a unique* $n \times n$ *invertible matrix* $P_{v \leftarrow u}$ *such that*

$$[\mathbf{x}]_v = P_{v \leftarrow u}[\mathbf{x}]_u,$$

for all $\mathbf{x} \in V$*. Furthermore, the matrix* $P_{v \leftarrow u}$ *has the form*

$$P_{v \leftarrow u} = [[\mathbf{u}_1]_v, \cdots, [\mathbf{u}_n]_v] .$$

Proof of Theorem 3: Both sets $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ and $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ are basis of V, then there exist a unique set of numbers $\{u_1, \cdots, u_n\}$ and $\{v_1, \cdots, v_n\}$ such that

$$\mathbf{x} = u_1\mathbf{u}_1 + \cdots + u_n\mathbf{u}_n = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}_u , \quad \mathbf{x} = v_1\mathbf{v}_1 + \cdots + v_n\mathbf{v}_n = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}_v .$$

Therefore,

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}_v = [[\mathbf{u}_1]_v, \cdots, [\mathbf{u}_n]_v] \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}_u .$$

This system of equations for $(u_1, \cdots u_n)$ has a unique solution solutions for all $(v_1, \cdots v_n)$, because the $\mathbf{u}$'s and $\mathbf{v}$'s are basis. That is, $P_{v \leftarrow u} = [[\mathbf{u}_1]_v, \cdots, [\mathbf{u}_n]_v]$ is invertible. □

Slide 6

Exercises

- (2, Sec. 4.7) Let $\{\mathbf{b}_1, \mathbf{b}_2\}$, $\{\mathbf{c}_1, \mathbf{c}_2\}$ be basis of $I\!R^2$. Let $\mathbf{b}_1 = -\mathbf{c}_1 + 4\mathbf{c}_2$ and $\mathbf{b}_2 = 5\mathbf{c}_1 - 3\mathbf{c}_2$.
  - Find $[\mathbf{x}]_c$ for $[\mathbf{x}]_b = (5, 3)_b$.
  - Find $[\mathbf{x}]_b$ for $[\mathbf{x}]_c = (1, 1)_c$.

  In P_2 find the change of coordinate matrix from the basis $B = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + t^2\}$. to the standard basis $\{1, t, t^2\}$. Find the B-coordinates of $\mathbf{x} = 1 - 2t$.

Slide 7

Determinants

- Determinants of 2×2 matrices.
- Definition.
- Properties.

Slide 8

2×2 determinant

Definition 4 *The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by*

$$\Delta = \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The determinant appears in the computation of the inverse matrix.

Slide 9

Properties

Theorem 4 (Main properties of 2×2 determinants)
Let $A = [\mathbf{a}_1, \mathbf{a}_2]$ be a 2×2 matrix. Let $\mathbf{c}$ be a 2-vector.

- $\det([\mathbf{a}_1 + \mathbf{c}, \mathbf{a}_2]) = \det([\mathbf{a}_1, \mathbf{a}_2]) + \det([\mathbf{c}, \mathbf{a}_2])$.
- $\det([c\mathbf{a}_1, \mathbf{a}_2]) = c\det([\mathbf{a}_1, \mathbf{a}_2])$.
- $\det([\mathbf{a}_1, \mathbf{a}_2]) = -\det([\mathbf{a}_2, \mathbf{a}_1])$.
- $\det([\mathbf{a}_1, \mathbf{a}_1]) = 0$.
- $\mathbf{a}_1$, $\mathbf{a}_2$ *are l.d.* $\Leftrightarrow \det([\mathbf{a}_1, \mathbf{a}_2]) = 0$.
- *A is invertible* $\Leftrightarrow \det(A) \neq 0$.
- $\det(A) = \det(A^T)$.

Slide 10

Properties

Theorem 5 (Determinants and elementary row operations)
Let A be a 2×2 matrix.

- *Let B be the result of adding to a row in A a multiple of another row in A. Then,* $\det(B) = \det(A)$.
- *Let B be the result of interchanging two rows in A. Then,* $\det(B) = -\det(A)$.
- *Let B be the result of multiply a row in A by a number k. Then,* $\det(B) = k\det(A)$.

Slide 11

Determinants and areas

Theorem 6 *Let $A = [\mathbf{a}_1, \mathbf{a}_2]$ be a 2×2 matrix, with $\mathbf{a}_1$ and $\mathbf{a}_2$ being nonzero and noncollinear. Then,* $|\det([\mathbf{a}_1, \mathbf{a}_2])|$ *is the area of the parallelogram formed by $\mathbf{a}_1$ and $\mathbf{a}_2$.*

Proof: Choose a basis $\mathbf{e}_1$, $\mathbf{e}_2$ such that $\mathbf{a}_1 = b\mathbf{e}_1$, for some number $b \neq 0$. Because $\mathbf{a}_1$ is not collinear to $\mathbf{a}_2$, there exists a $c \neq 0$ such that the vector $\mathbf{u} = c\mathbf{a}_1 + \mathbf{a}_2$ is collinear to $\mathbf{e}_2$. For that vector $\mathbf{u}$ holds that $\mathbf{u} = h\mathbf{e}_2$, where $|h|$ is the height of the parallelogram. Summarizing:

$$\mathbf{a}_1 = \begin{bmatrix} b \\ 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 0 \\ h \end{bmatrix}.$$

Now, the determinant of A is the following:

$$\det(A) = \det([\mathbf{a}_1, \mathbf{a}_2]) = \det([\mathbf{a}_1, \mathbf{a}_2 + c\mathbf{a}_1]) = \det([\mathbf{a}_1, \mathbf{u}]).$$

Slide 12

Therefore,

$$\det(A) = \begin{vmatrix} b & 0 \\ 0 & h \end{vmatrix} = hb.$$

Then, $|\det(A)| = |h|\,|b|$, where $|h|$ is the height and $|b|$ the length of the base of the parallelogram. $\square$.

Slide 13

Determinants

- Determinant of 3×3 matrices.
- Determinant of $n \times n$ matrices.
- Some properties.

Slide 14

Determinant of 3×3 *matrices*

Definition 5 *The determinant of a* 3×3 *matrix* A *is given by*

$$\begin{aligned}
\det(A) &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} a_{11} - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} a_{12} + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} a_{13}, \\
&= (a_{22}a_{33} - a_{32}a_{23})a_{11} - (a_{21}a_{33} - a_{31}a_{23})a_{12} \\
&\quad +(a_{21}a_{32} - a_{31}a_{22})a_{13}.
\end{aligned}$$

This formula is called "expansion by the first row."

Slide 15

Determinant of 3×3 *matrices*

Theorem 7 (Expansions by rows) *The determinant of a* 3×3 *matrix* A *can also be computed with an expansion by the second row or by the third row.*

The proof is just do the calculation. For example, the expansion by the second row is the following:

$$-\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} a_{21} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} a_{22} - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} a_{23}$$

$$= -(a_{12}a_{33} - a_{32}a_{13})a_{21} + (a_{11}a_{33} - a_{31}a_{13})a_{22} - (a_{11}a_{32} - a_{31}a_{12})a_{23}.$$

$$= \det(A).$$

Slide 16

Determinant of 3×3 *matrices*

Theorem 8 (Expansions by columns) *The determinant of a* 3×3 *matrix* A *can also be computed with an expansion by the any of its columns.*

The proof is again to do the calculation. For example, the expansion by the first column is the following:

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} a_{11} - \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} a_{21} + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} a_{31}$$

$$= (a_{22}a_{33} - a_{32}a_{23})a_{11} - (a_{12}a_{33} - a_{32}a_{13})a_{21} + (a_{12}a_{23} - a_{22}a_{13})a_{31}.$$

$$= \det(A).$$

Slide 17

Determinant of $n \times n$ *matrices*

Notation:

$$A_{ij} = \begin{bmatrix} a_{11} & \cdots & \mathbf{a_{1j}} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ \mathbf{a_{i1}} & \cdots & \mathbf{a_{ij}} & \cdots & \mathbf{a_{in}} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & \mathbf{a_{nj}} & \cdots & a_{nn} \end{bmatrix}$$

$$A = \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$ Sign of coefficient a_{ij} is $(-1)^{i+j}$.

Slide 18

Determinant of $n \times n$ matrices

Definition 6 *The determinant of an $n \times n$ matrix $A = [a_{ij}$ is given by*

$$\begin{aligned} \det(A) &= \det(A_{11})a_{11} - \det(A_{12})a_{12} + \cdots + (-1)^{1+n}\det(A_{1n})a_{1n}, \\ &= \sum_{j=1}^{n}(-1)^{1+j}\det(A_{1j})a_{1j}. \end{aligned}$$

This formula is called "expansion by the first row."

Slide 19

Determinant of $n \times n$ matrices

Theorem 9 *The determinant of an $n \times n$ matrix $A = [a_{ij}]$ can be computed by an expansion along any row or along any column. That is,*

$$\begin{aligned} \det(A) &= \sum_{j=1}^{n}(-1)^{i+j}\det(A_{ij})a_{ij}, \quad \textit{for any } i = 1, \cdots, n, \\ &= \sum_{i=1}^{n}(-1)^{i+j}\det(A_{ij})a_{ij}, \quad \textit{for any } j = 1, \cdots, n. \end{aligned}$$

Notation: The cofactor C_{ij} of a matrix A is the number given by

$$C_{ij} = (-1)^{i+j}\det(A_{ij}).$$

Theorem 10 $\det(A) = \det(A^T)$.

Slide 20

Determinant of $n \times n$ matrices

Suggestion: If a matrix has a row or a column with several zeros, then it is simpler to compute its determinant by an expansion along that row or column.

Theorem 11 *The determinant of a triangular matrix is the product of its diagonal elements.*

Examples:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} (1) = 1 \times 4 \times 6 = 24.$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 1 & 5 \end{vmatrix} (1) = 1 \times 3 \times 5 = 15.$$