Slide 1

Basis and dimensions

- Review: Subspace of a vector space. (Sec. 4.1)
- Linear combinations, l.d., l.i. vectors. (Sec. 4.3)
- Dimension and Base of a vector space. (Sec. 4.4)

Slide 2

Review: Vector space

A vector space is a set of elements of any kind, called vectors, on which certain operations, called addition and multiplication by numbers, can be performed.

The main idea in the definition of vector space is to do not specify the nature of the elements nor do we tell how the operations are to be performed on them. Instead, we require that the operations have certain properties, which we take as axioms of a vector space.

Examples include spaces of arrows, matrices, functions.

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Review: Subspace

Definition 1 (Subspace) *A subspace W of a vector space V is a subset of V that is closed under the addition and scalar multiplication operations on V.*

That is, $W \subset V$, and for all $\mathbf{u}$, $\mathbf{v} \in W$ and $a \in I\!R$ holds that

$$\mathbf{u} + \mathbf{v} \in W, \quad a\mathbf{u} \in W.$$

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Examples

- The set $W \subset I\!R^3$ given by

  $$W = \{\mathbf{x} \in I\!R^3 : \mathbf{x} = (x_1, x_2, 0), \quad x_1, x_2 \in I\!R\},$$

  is a subspace of $I\!R^3$.
- The set $\widehat{W} \subset I\!R^3$ given by

  $$\widehat{W} = \{\mathbf{x} \in I\!R^3 : \mathbf{x} = (x_1, x_2, 1), \quad x_1, x_2 \in I\!R\},$$

  in contrast is not a subspace of $I\!R^3$.

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Examples

- The set $W = \{(x_1, x_2) \in I\!R^2 : x_1 \geq 0\}$ is not a subspace of $I\!R^2$.
- The segment $W\{x \in I\!R : -1 \leq x \leq 1\}$ is not a subspace of $I\!R$.
- The line $W = \{\mathbf{x} \in I\!R^3 : \mathbf{x} = (1, 2, 3)t\}$ is a subspace or $I\!R^3$.
- The line $W = \{\mathbf{x} \in I\!R^3 : \mathbf{x} = (1, 2, 0) + (1, 2, 3)t\}$ is not a subspace or $I\!R^3$.

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Linear combinations

Definition 2 (Linear combination) *Let V be a vector space, $\mathbf{v}_1, \cdots, \mathbf{v}_k \in V$ be arbitrary vectors, and $a_1, \cdots, a_k \in I\!R$ be arbitrary scalars. We call a linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_k$ the vector*

$$\sum_{i=1}^{k} a_i \mathbf{v}_i = a_1 \mathbf{v}_1 + \cdots + a_k \mathbf{v}_k.$$

Linear combinations give rise to the concept of span of a set of vectors.

Definition 3 (Span) *The span of $\mathbf{v}_1, \cdots, \mathbf{v}_k \in V$ is the set of all linear combinations of these vectors, that is,*

$$Span\{\mathbf{v}_1, \cdots, \mathbf{v}_k\} = \left\{ \sum_{i=1}^{k} a_i \mathbf{v}_i \;:\; a_i \in I\!R,\; 0 \leq i \leq k \right\}.$$

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Notice that every linear combination of $\mathbf{v}_1, \cdots, \mathbf{v}_k$ also belongs to V, so $\text{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_k\} \subset V$. One can also check that it is a subspace.

Claim 1 *Let V be a vector space and $\mathbf{v}_1, \cdots, \mathbf{v}_k \in V$. Then, Span$\{\mathbf{v}_1, \cdots, \mathbf{v}_k\} \subset V$ is a subspace.*

Different sets may span the same subspace.

Example:

- The space $I\!R^2$ is spanned by $\{\mathbf{e}_1, \mathbf{e}_2\}$, and also by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + 2\mathbf{e}_2\}$, and also by $\{\mathbf{0}, \mathbf{e}_1 + 2\mathbf{e}_2, -\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2\}$.
- The space P_2 is spanned by $\{1, t, t^2\}$, and also by $\{1, t, t^2, (t^2 - 1), (t + 1)^2\}$.

It is then useful to remove redundant vectors from linear combinations.

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Definition 4 (l.d.) *Let V be a vector space. The vectors $\mathbf{v}_1, \cdots, \mathbf{v}_k \in V$ are linearly dependent, l.d., if there exist scalars $a_1, \cdots, a_k \in R$ with at least one of them nonzero, such that*

$$\sum_{i=1}^{k} a_i \mathbf{v}_i = 0.$$

Definition 5 (l.i.) *Let V be a vector space. The vectors $\mathbf{v}_1, \cdots, \mathbf{v}_k \in V$ are linearly independent, l.i., if they are not l.d., that is, the only choice of scalars $a_1, \cdots, a_k \in R$ for which the equation*

$$\sum_{i=1}^{k} a_i \mathbf{v}_i = 0$$

holds is $a_i = 0$ for all $1 \leq i \leq k$.

Linearly independent set of vectors contain no redundant vectors, in the sense of linear combinations.

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Examples:

- If one element in $S \subset V$ is a scalar multiple of another, S is l.d..
- If $\mathbf{0} \in S$, then S is l.d..
- Let V be the space of continuous functions on $[0, 2\pi]$. The set of vectors $\mathbf{v}_1 = \cos^2(t)$, $\mathbf{v}_2 = \sin^2(t)$, $\mathbf{v}_3 = 1$ is l.d.. Indeed, Pythagoras theorem says that $\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = 0$.
- Consider the space P_2. The set $\{1, t, t^2\}$ is l.i.. Indeed, the equation
$$a_0 + a_1 t + a_2 t^2 = 0,$$
implies that all coefficient vanishes. Evaluate the equation at $t = 0$, then $a_0 = 0$. So we have $a_1 t + a_2 t^2 = 0$. If the function vanishes, its derivative also vanishes. So $a_1 + a_2 t = 0$. Evaluate this at $t = 0$, then $a_1 = 0$. Repeat the procedure one more time, then $a_2 = 0$.

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Dimensions and base

Definition 6 (Finite dimension and base)

- *A vector space V has finite dimension is there exists a maximal set of l.i. vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$.*
- *The number n is called the dimension of V, and we denote by $n = \dim V$.*
- *The l.i. vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ are called a base of V.*
- *If there is no maximal set of l.i. vectors, then V is called infinite dimensional.*

That is, the set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is maximal if it is l.i. but the set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n, \mathbf{v}_{n+1}\}$ is l.d. for all $\mathbf{v}_{n+1} \in V$.

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Examples:

- The set $\{\mathbf{e}_1, \mathbf{e}_2\} \in I\!R^2$ is a basis. Another basis is $\{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2\}$. The set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$ is not a basis. The set $\{\mathbf{e}_1, 2\mathbf{e}_1\}$ is not a basis.
- Notice that a basis of a vector space is not unique.
- The space $I\!R^n$ is finite dimensional, of dimension n, because the vectors $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ are l.i., and any set of $n+1$ vectors in $I\!R^n$ is l.d..
- The space P, polynomials on $[0,1]$ is infinite dimensional. The infinite set $\{1, t, t^2, t^3, \cdots\}$ is l.i..

Theorem 1 *A set of vectors $E \subset V$ is a base if the vectors in E are l.i. and span V.*

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Components or coordinates

Theorem 2 *Let V be a finite dimensional vector space with $n = \dim V$. If $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ is a basis of V, then each vector $\mathbf{v} \in V$ has a unique decomposition*

$$\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{u}_i.$$

The n scalars a_i are called components or coordinates of $\mathbf{v}$ with respect to this basis.

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Exercises

Let $\mathbf{u}_1 = (1,1)$ and $\mathbf{u}_2 = (1,-1)$, be a basis of $I\!R^2$. Find the components of $\mathbf{x} = (1,2)$ in that basis.

The vectors $\mathbf{u}_1$, $\mathbf{u}_2$ form a basis so there exists constants c_1, c_2 such that

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2.$$

That is,

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then one has to solve the augmented matrix

$$\begin{bmatrix} 1 & 1 & | & 1 \\ 1 & -1 & | & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & \frac{3}{2} \\ 0 & 1 & | & -\frac{1}{2} \end{bmatrix},$$

so $c_1 = 3/2$ and $c_2 = -1/2$, and then $\mathbf{x} = 3/2\mathbf{u}_1 - 1/2\mathbf{u}_2$.

Proof: The set $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ is a basis so there exists scalars a_i, for $1 \leq i \leq n$ such that the following decomposition holds,

$$\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{u}_i.$$

This decomposition is unique. Because, if there is another decomposition

$$\mathbf{v} = \sum_{i=1}^{n} b_i \mathbf{u}_i.$$

then the difference has the form

$$\sum_{i=1}^{n} (a_i - b_i)\mathbf{u}_i = 0.$$

Because the vectors $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ are l.i. this implies that $a_i = b_i$ for all $0 \leq i \leq n$. □

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Null and Column spaces

- Null space and range. (Sec. 4.2)
- Examples.
- Linear transformations.
- Null space and range for L.T..
- Fundamental theorem of algebra. (Sec. 4.6)

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Null and range spaces: Matrix version

Definition 7 *Let A be an $m \times n$. The null space of A, denoted as $N(A) \subset V$, is the set of all elements of V solution of $A\mathbf{v} = \mathbf{0}$, that is,*

$$N(A) = \{\mathbf{v} \in V \ : \ A\mathbf{v} = \mathbf{0}\}.$$

Definition 8 *Let A be an $m \times n$. The column space of A, denoted as $Col(A)$, is the span of the columns of A. Its dimension is called the rank of A, that is* $\mathrm{rank}(A) = \dim Col(A)$.

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Exercises

Find the null space of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

The null space are all $\mathbf{x} \in \mathbb{R}^3$ solutions of $A\mathbf{x} = \mathbf{0}$. One can check that these solutions have the form

$$\mathbf{e} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} x_3.$$

Therefore, $N(A) = \text{Span}\{(-3, 1, 1)\}$.

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Exercises

(Problem 20, Sec. 4.2) Find k_1 and k_2 such that $N(A)$ is a subspace of $\mathbb{R}^{k_1}$, and $\text{Col}(A)$ is a subspace of $\mathbb{R}^{k_2}$, with $A = [1, -3, 9, 0, -5]$.

The solutions $\mathbf{x}$ of the equation $A\mathbf{x} = \mathbf{0}$ form a hyperplane in $\mathbb{R}^5$, given by the equation

$$x_1 - 3x_2 + 9x_3 - 5x_5 = 0.$$

So, the solutions can be written as

$$\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -9 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_5.$$

$N(A) = \text{Span}\{(3, 1, 0, 0, 0), (-9, 0, 1, 0, 0), (0, 0, 0, 1, 0), (5, 0, 0, 0, 1)\} \subset \mathbb{R}^4$.

$\text{Col}(A) \subset \mathbb{R}$.

linear transformations

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Definition 9 *Let V, W be vector spaces. The function $T : V \to W$ is called a linear transformation if it has the following properties:*

- $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ *for all* $\mathbf{v}_1$, $\mathbf{v}_2 \in V$*;*
- $T(a\mathbf{v}) = aT(\mathbf{v})$*, for all* $\mathbf{v} \in V$ *and* $a \in \mathbb{R}$*.*

These properties means that T preserves addition and multiplication by numbers. The two properties combine in one formula

$$T(a\mathbf{v} + b\mathbf{u}) = aT(\mathbf{v}) + bT(\mathbf{u}),$$

or equivalently,

$$T\left(\sum_{i=1}^{k} a_i \mathbf{v}_i\right) = \sum_{i=1}^{k} a_i T(\mathbf{v}_i).$$

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Examples:

- The identity transformation, that is $T : V \to V$, given by $T(\mathbf{v}) = \mathbf{v}$.
- A stretching by $a \in \mathbb{R}$, that is, $T : V \to V$, given by $T(\mathbf{v}) = a\mathbf{v}$.
- A projection, that is $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$.
- The derivative. Let V be the vector space of differentiable functions on $(0, 1)$, and let W be the vector space of continuous function on $(0, 1)$. Then, $T : V \to W$ given by $T(f) = f'$ is a linear transformation.

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Null space and range of T

Definition 10 (Null space of T) *Let $T : V \to W$ be a linear transformation. The null space of T, denoted as $N(T) \subset V$, is the set of all elements of V that T maps onto the $\mathbf{0} \in W$. That is,*

$$N(T) = \{\mathbf{v} \in V \ : \ T(\mathbf{v}) = \mathbf{0}\}.$$

Definition 11 (Range of T) *Let $T : V \to W$ be a linear transformation. The range of T, denoted as $T(V) \subset W$, is the set of all elements of W of the form $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v} \in V$.*

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Theorem 3 ($N(T)$ subspace of V) *Let $T : V \to W$ be a linear transformation. The null space of T is a subspace of V.*

Theorem 4 ($T(V)$ subspace of W) *Let $T : V \to W$ be a linear transformation. The range of T, denoted by $T(V) \subset W$, is a subspace of W.*

The nullity is the dimension of $N(T)$.

The rank is the dimension of $T(V)$.

Theorem 5 *Let T be the linear transformation associated to A. Then $N(T) = N(A)$ and $T(V) = Col(A)$.*

Proof of Theorem 3: Assume that $\mathbf{u}$ and $\mathbf{v} \in N(T)$. Then, $a\mathbf{u} + b\mathbf{v} \in N(T)$ because

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v}) = a\mathbf{0} + b\mathbf{0} = \mathbf{0}.$$

□

Proof of Theorem 4: Given two vectors $T(\mathbf{u})$ and $T(\mathbf{v})$ in the range of T, then $aT(\mathbf{u}) + bT(\mathbf{v})$ also belongs to the range of T because T is linear, that is,

$$aT(\mathbf{u}) + bT(\mathbf{v}) = T(a\mathbf{u} + b\mathbf{v}).$$

□

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Examples:

- The identity transformation. The null space is $N(I) = \{\mathbf{0}\}$. The range space is $T(V) = V$.
- A stretching by $a \in I\!R$. In the case $a \neq 0$ the null space is $N(T) = \{\mathbf{0}\}$ and $T(V) = V$. In the case $a = 0$ one has $N(T) = V$ and $T(V) = \{\mathbf{0}\}$.
- A projection, $T : I\!R^3 \to I\!R^3$ given by $T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$. Then $N(T) = \{x_3\mathbf{e}_3\}$. The range space is the plane spanned by $\{\mathbf{e}_1, \mathbf{e}_2\}$.
- The derivative, given by $T(f) = f'$. Then $N(T)$ are the constant functions. The range of T are the continuous functions on $(0, 1)$.

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Fundamental theorem of algebra

Theorem 6 (Nullity plus rank) *Let $T : V \to W$ be a linear transformation, and V be finite dimensional. Then $N(T)$ and $T(V)$ are finite dimensional and the following relation holds,*

$$\dim N(T) + \dim T(V) = \dim V.$$

Theorem 7 (Nullity plus rank: Matrix form) *Let A be an $m \times n$ matrix with m, n finite Then, the following relation holds,*

$$\dim N(A) + \operatorname{rank}(A) = n.$$

Proof of Theorem 6: Let $n = \dim V$ and let $S = \{\mathbf{e}_1, \cdots, \mathbf{e}_k\}$ be a basis for $N(T)$, so we say that the nullity is some number $k \geq 0$. Because $N(T)$ is contained in V one knows that $k \leq n$. Let us add l.i. vectors to S to complete a basis of V, say, $\{\mathbf{e}_1, \cdots, \mathbf{e}_k, \mathbf{e}_{k+1}, \cdots, \mathbf{e}_{k+r}\}$ for some number $r \geq 0$ such that $k + r = n$. We shall prove that $\{T(\mathbf{e}_{k+1}), \cdots, T(\mathbf{e}_{k+r})\}$ is a basis for $T(V)$, and then $r = \dim T(V)$. This relation proves Theorem 6.

The elements $\{T(\mathbf{e}_{k+1}), \cdots, T(\mathbf{e}_{k+r})\}$ are a basis of $T(V)$ because they span $T(V)$ and they are l.i.. They span $T(V)$ because for every $\mathbf{w} \in T(V)$ we know that there exists $\mathbf{v} \in V$ such that $\mathbf{w} = T(\mathbf{v})$. If we write $\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i$, then we have

$$\mathbf{w} = T\left(\sum_{i=0}^{n} \mathbf{e}_i\right) = \sum_{i=0}^{n} a_i T(\mathbf{e}_i) = \sum_{i=k+1}^{k+r} a_i T(\mathbf{e}_i),$$

then the $\{T(\mathbf{e}_{k+1}), \cdots, T(\mathbf{e}_{k+r})\}$ span $T(V)$.

These vectors are also l.i., by the following argument. Suppose there are scalars $c_{k+1}, \cdots, c_{k+r}$ such that

$$\sum_{i=k+1}^{k+r} c_i T(\mathbf{e}_i) = 0.$$

Then, this implies

$$T\left(\sum_{i=k+1}^{k+r} c_i \mathbf{e}_i\right) = 0,$$

so the vector $\mathbf{u} = \sum_{i=k+1}^{k+r} c_i T(\mathbf{e}_i)$ belongs to $N(T)$. But if $\mathbf{u}$ belongs to $N(T)$, then it must be written also as a linear combination of the elements of the base of $N(T)$, namely, the vectors $\mathbf{e}_1, \cdots, \mathbf{e}_k$, so there exists constants $c_1, \cdots, c_k$ such that

$$\mathbf{u} = \sum_{i=1}^{k} c_i \mathbf{e}_i.$$

Then, we can construct the linear combination

$$\mathbf{0} = \mathbf{u} - \mathbf{u} = \sum_{i=1}^{k} c_i \mathbf{e}_i - \sum_{i=k+1}^{k+r} c_i \mathbf{e}_i.$$

Because the set $\{\mathbf{e}_1 \cdots, \mathbf{e}_{k+r}\}$ is a basis o V we have that all the c_i with $1 \leq i \leq k+r$ must vanish. Then, the vectors $\{T(\mathbf{e}_{k+1}), \cdots, T(\mathbf{e}_{k+r})\}$ are l.i.. Therefore they are a basis of $T(V)$, and then the dimension of $\dim T(V) = r$. This proves the Theorem. $\square$

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The matrix of a linear transformation

Theorem 8 *Let $T : V \to W$ be a linear transformation, where V and W are finite dimensional vector spaces with $\dim V = n$ and $\dim W = m$. Let $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be a basis of V, and $\{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$ be a basis of W.*

Then, T has associated a matrix $A = [\mathbf{a}_i]$, $1 \leq i \leq n$, given by

$$T(\mathbf{v}_i) = \mathbf{a}_i = \sum_{j=1}^{m} a_{ij} \mathbf{w}_j.$$

Idea of the proof: every vector in W is a linear combinations of the basis vectors $\mathbf{w}_j$ for $0 \leq j \leq m$. In particular all the n vectors $T(\mathbf{e}_i)$, $0 \leq i \leq n$. therefore, there must exist the $m \times n$ set of numbers a_{ij} given above. These coefficients define the matrix A associated to T.

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Example:

- In the case $T : \mathbb{R}^2 \to \mathbb{R}^3$, one chooses the basis vectors $\mathbf{v}_1 = (1,0)$, $\mathbf{v}_2 = (0,1)$, for $\mathbb{R}^2$, and the basis vectors $\mathbf{w}_1 = (1,0,0)$, $\mathbf{w}_2 = (0,1,0)$, $\mathbf{w}_3 = (0,0,1)$ for $\mathbb{R}^3$. Then, a linear transformation

$$T(x_1, x_2) = (x_1 + 3x_2, -x_1 + x_2, x_2),$$

has associated the matrix $A = [\mathbf{a}_1, \mathbf{a}_2]$ given by

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

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Invertible transformations

- One-to-one and onto.
- Isomorphisms (one-to-one and onto).
- Invertible transformations.

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One-to-one and onto

Definition 12 (one-to-one onto) *Let $T : V \to W$ be a linear transformation.*

- *T is one-to-one if $\mathbf{v}_1 \neq \mathbf{v}_2$ implies $T(\mathbf{v}_1) \neq T(\mathbf{v}_2)$;*
- *T is onto if for all $\mathbf{w} \in W$ there exists $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$.*

Theorem 9 *Let $T : V \to W$ be a linear transformation.*

- *T is one-to-one $\Leftrightarrow N(T) = \{\mathbf{0}\}$;*
- *T is onto $\Leftrightarrow T(V) = W$.*

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Isomorphism

Definition 13 (Isomorphism) *The linear transformation $T : V \to W$ is an isomorphism if T is one-to-one and onto.*

Isomorphic spaces are essentially the same. There is a one to one correspondence between elements from one space and of the other.

Example:

- P_2 is isomorphic to R^3. A basis of P_2 is $\{1, t, t^2\}$ A general element $\mathbf{v} \in P$ has the form $\mathbf{v} = a_0 + a_1 t + a_2 t^2$. The linear transformation $T : P_2 \to R^3$ given by $T(a_0 + a_1 t + a_2 t^2) = a_0 \mathbf{e}_0 + a_1 \mathbf{e}_2 + a_2 \mathbf{e}_3 \in I\!R^3$ is an isomorphism.

Theorem 10 *Let $T : V \to W$ be an isomorphism, with V and W of finite dimension. Then* $\dim V = \dim W$.