Slide 1

(Non) Homogeneous systems

- Definition
- Examples
- Read Sec. 1.6

Slide 2

(Non) Homogeneous systems

Definition 1 *A linear system of equations*

$$A\mathbf{x} = \mathbf{b}$$

is called homogeneous if $\mathbf{b} = 0$, *and non-homogeneous if* $\mathbf{b} \neq 0$.

Notice that $\mathbf{x} = 0$ is always solution of the homogeneous equation.

The solutions of an homogeneous system with 1 and 2 free variables are a lines and a planes, respectively, *through the origin.*

Slide 3

Theorem 1 *Let A, $\mathbf{b}$ be an $m \times n$ matrix and an m vector, respectively. Assume that the system $A\mathbf{x} = \mathbf{b}$ is consistent, and let $\mathbf{x}_0$ be one solution.*

Then, every solution $\mathbf{x}$ can be written as

$$\mathbf{x} = \mathbf{x}_h + \mathbf{x}_0,$$

where $\mathbf{x}_h$ is a solution of the homogeneous system, that is,

$$A\mathbf{x}_h = 0.$$

Solutions of the non-homogeneous equation $A\mathbf{x} = \mathbf{b}$ are obtained translating solutions of the homogeneous equation $A\mathbf{x}_h = 0$, using a particular solution $\mathbf{x}_0$ of the non-homogeneous system.

Slide 4

Recall that a vector $\mathbf{x} \in I\!R^3$ belongs to a plane passing through the origin with normal vector $\mathbf{n}$ if $\mathbf{x} \cdot \mathbf{n} = 0$, where '$\cdot$' is the dot product of vectors. This is the vector equation of the plane. In coordinates,

$$n_1 x_1 + n_2 x_2 + n_3 x_3 = 0.$$

Suppose that $n_1 \neq 0$. Then, x_2 and x_3 are free variables, and

$$x_1 = -\frac{n_2}{n_1} x_2 - \frac{n_3}{n_1} x_3.$$

The parametric equation of the plane is given by

$$\mathbf{x} = \mathbf{v} x_2 + \mathbf{w} x_3,$$

where

$$\mathbf{v} = \begin{pmatrix} -\frac{n_2}{n_1} \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -\frac{n_3}{n_1} \\ 0 \\ 1 \end{pmatrix}.$$

Slide 5

Linear (in)dependent vectors

Definition 2 (l.d.) *A set of vectors* $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$*, with* $k \geq 1$ *is called linearly dependent, denoted by l.d., if one of them can be written as a linear combination of the others.*

That is, there exist constants $a_1, \cdots, a_k$, at least one of them nonzero, such that,

$$a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k = 0.$$

Definition 3 (l.i.) *A set of vectors* $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$*, with* $k \geq 1$ *is called linearly independent, denoted by l.i., if it is not l.d..*

That is, none of them can be written as a linear combination of the others. That is,

$$a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k = 0 \Rightarrow a_1 = a_2 = \cdots = a_k = 0.$$

Slide 6

Linear (in)dependent vectors

Note: Any set of vectors containing the zero vector $\mathbf{0}$ is l.d..
(Proof: Take $\mathbf{v}_1 = \mathbf{0}$, then choose $a_1 \neq 0$, and all the others $a_2 = \cdots = a_k = 0$. Then,

$$a_1\mathbf{0} = 0v_2 + \cdots + 0\mathbf{v}_k.)$$

These definitions try to capture the idea that two noncollinear vectors in a plane are not multiple of each other (that is, they are are l.i.). And if you incorporate a third vector in that plane then, any of the three vectors can be written as a linear combination of the other two (that is, the resulting three vectors are l.d.).

So, l.d. vectors means that there is at least one redundant vector, in this sense of linear combinations.

Slide 7

Linear (in)dependent vectors

Theorem 2 *The vectors* $\{\mathbf{a}_1, \cdots, \mathbf{a}_k\}$ *are l.i.* $\Leftrightarrow$ *The system* $A\mathbf{x} = 0$ *with* $A = [\mathbf{a}_1, \cdots, \mathbf{a}_k]$ *has* **only** *the trivial* $\mathbf{x} = \mathbf{0}$ *solution.*

This result can also be written as follows:

$\{\mathbf{a}_1, \cdots, \mathbf{a}_k\}$ are l.d. $\Leftrightarrow$ $A\mathbf{x} = 0$ has a nontrivial solution.

Slide 8

Linear transformations

- Review of linear functions.
- Definition of linear transformations.
- Examples.

Slide 9

Review of linear functions

The function $y : I\!R \to I\!R$ given by $y = ax + c$ is linear in the sense that x appears linearly.

A generalization $\mathbf{y} : I\!R^n \to I\!R^m$ is given by $\mathbf{y} = A\mathbf{x} + \mathbf{c}$, where $\mathbf{x}$ is an n vector, A is an $m \times n$ matrix, and $\mathbf{y}$, $\mathbf{c}$ are m vectors.

Consider the case $\mathbf{c} = 0$, that is, $\mathbf{y} = A\mathbf{x}$. Then,

$$\begin{aligned} A(\mathbf{x}_1 + \mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2 \quad \Rightarrow \quad \mathbf{y}(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{y}(\mathbf{x}_1) + \mathbf{y}(x_2), \\ A(a\mathbf{x}) &= aA\mathbf{x} \quad \Rightarrow \quad \mathbf{y}(a\mathbf{x}) = a\mathbf{y}(\mathbf{x}). \end{aligned}$$

Converse is true for $1 \leq m < \infty$, and $1 \leq n < \infty$.

Slide 10

Review of linear functions

Theorem 3 *Let $\mathbf{y} : I\!R^n \to I\!R^m$, with $1 \leq m < \infty$, and $1 \leq n < \infty$, such that it satisfies the properties*

$$\begin{aligned} \mathbf{y}(\mathbf{x}_1 + \mathbf{x}_2) &= \mathbf{y}(\mathbf{x}_1) + \mathbf{y}(x_2), \\ \mathbf{y}(a\mathbf{x}) &= a\mathbf{y}(\mathbf{x}). \end{aligned}$$

Then, there exists an $m \times n$ matrix A such that

$$\mathbf{y} = A\mathbf{x}.$$

Change names, $\mathbf{y} \to T$.

Slide 11

Linear transformations

Definition 4 *A function $T : I\!R^n \to I\!R^m$ is called a linear transformation if*

$$\begin{aligned} T(\mathbf{x}_1 + \mathbf{x}_2) &= T(\mathbf{x}_1) + T(x_2), && \forall\, \mathbf{x}_1, \mathbf{x}_2 \in I\!R^n, \\ T(a\mathbf{x}) &= aT(\mathbf{x}), && \forall\, \mathbf{x} \in I\!R^n. \end{aligned}$$

$I\!R^n$ is called the domain of T. $I\!R^m$ is called the codomain of T. and $T(I\!R^n)$ is called the range of T, where

$$T(I\!R^n) = \{\mathbf{v} \in I\!R^m : \text{exists } \mathbf{x} \in I\!R^n, \text{ such that } T(\mathbf{x}) = \mathbf{v}\}.$$

Slide 12

Example: $T : I\!R^2 \to I\!R^3$,

$$T\mathbf{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} x_2,$$

so $T(I\!R^2) = I\!R^2$, that is, the range of T is $I\!R^2$, a subset of the codomain, R^3.

Theorem 4 *Let $T : I\!R^n \to I\!R^m$ be a linear transformation, with $1 \leq m < \infty$, and $1 \leq n < \infty$. Then, there exists an $m \times n$ matrix A such that*

$$T(\mathbf{x}) = A\mathbf{x}.$$