Slide 1

System of Linear Equations

Definition 1 *Fix a set of numbers a_{ij}, b_i, where $i = 1, \cdots, m$ and $j = 1, \cdots, n$. A system of m linear equations in n variables x_j, is given by*

$$\begin{array}{cccc} a_{11}x_1 + & \cdots & +a_{1n}x_n & = & b_1, \\ \vdots & & \vdots & & \vdots \\ a_{m1}x_1 + & \cdots & +a_{mn}x_n & = & b_m. \end{array}$$

Consistent: It has solutions (one or infinitely many).

Inconsistent: It has no solutions.

Slide 2

Solutions to a system of linear equations can be obtained by:

- Substitution. (Convenient for small systems.)
- Elementary Row Operations. (Convenient for large systems.)

It is not needed to write down the variable $x_1, \cdots, x_n$ while performing the EROs. Only the coefficients of the system, and the right hand side is needed.

This is the reason to introduce the matrix notation.

Slide 3

Matrix Notation

System of Equations: Augmented Matrix

$$\begin{array}{cccc} a_{11}x_1+ & \cdots & +a_{1n}x_n & = & b_1, \\ \vdots & & \vdots & & \vdots \\ a_{m1}x_1+ & \cdots & +a_{mn}x_n & = & b_m, \end{array} \rightarrow \begin{pmatrix} a_{11} & \cdots & a_{1n} & | & b_1, \\ \vdots & & \vdots & | & \vdots \\ a_{m1} & \cdots & a_{mn} & | & b_m, \end{pmatrix}$$

$$m \text{ rows} \overbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}^{n \text{ columns}}$$

$m \times n$ matrix.

Slide 4

Elementary Row Operations (EROs)

- Add to one row a multiple of the other.
- Interchange two rows.
- Multiply a row by a nonzero constant.

EROs do not change the solutions of a linear system of equations.

EROs are performed until the matrix is in echelon form.

Echelon form: Solutions of the linear system can be easily read out.

Slide 5

Definition 2 *The diagonal elements of a matrix*

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

are given by a_{ii}*, for* i *from* 1 *to the minimum of* m *and* n*.*

Examples: Only the diagonal elements are given.

$$\begin{pmatrix} a_{11} & * & * \\ * & a_{22} & * \\ * & * & a_{33} \end{pmatrix}, \quad \begin{pmatrix} a_{11} & * & * \\ * & a_{22} & * \end{pmatrix}, \quad \begin{pmatrix} a_{11} & * \\ * & a_{22} \\ * & * \end{pmatrix}.$$

Slide 6

Echelon Forms

- Echelon form: Upper triangular.
  (Every element below the diagonal is zero.)
- Reduced Echelon Form: A matrix in echelon form such that the first nonzero element in every row satisfies both,
  - it is equal to 1,
  - it is the only nonzero element in that column.

Slide 7

Existence and uniqueness

- A system of linear equations is *inconsistent* if and only if the echelon form of the augmented matrix has a row of the form

$$[0, \cdots, 0 | b \neq 0].$$

- A consistent system of linear equations contains either,
  - a unique solution, that is, *no* free variables;
  - or infinitely many solutions, that is, at least one free variable.

Slide 8

Vectors in $\mathbb{R}^n$

- Definition, Operations, Components.
- Linear combinations.
- Span.

Slide 9

Definition, Operations, Components

Definition 3 *A vector in* $\mathbb{R}^n$, $n \geq 1$, *is an oriented segment.*

Operations:

- Addition, Difference: Parallelogram law.
- Multiplication by a number: Stretching, compressing.

In components:

$$\mathbf{v} \pm \mathbf{w} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \pm \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 \pm w_1 \\ \vdots \\ v_n \pm w_n \end{pmatrix}, \quad a\mathbf{v} = \begin{pmatrix} av_1 \\ \vdots \\ av_n \end{pmatrix}.$$

Slide 10

Some properties of the addition and multiplication by a scalar:

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \mathbf{v} + \mathbf{u}, \\ \mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (\mathbf{u} + \mathbf{v}) + \mathbf{w}, \\ a(\mathbf{u} + \mathbf{v}) &= a\mathbf{u} + a\mathbf{v}, \\ (a + b)\mathbf{u} &= a\mathbf{u} + b\mathbf{u}. \end{aligned}$$

Slide 11

Definition 4 *A vector* $\mathbf{w} \in \mathbb{R}^n$ *is a linear combination of* $p \geq 1$ *vectors* $\mathbf{v}_1, \cdots, \mathbf{v}_p$ *in* $\mathbb{R}^n$ *if there exist* p *numbers* $c_1, \cdots, c_p$ *such that*

$$\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p.$$

A system of linear equations can be written as a vector equation:

$$\begin{array}{cccc} a_{11}x_1 + & \cdots & + a_{1n}x_n & = & b_1, \\ \vdots & & \vdots & & \vdots \\ a_{m1}x_1 + & \cdots & + a_{mn}x_n & = & b_m. \end{array}$$

$$\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} x_1 + \cdots + \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} x_n = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$\mathbf{a}_1\, x_1 + \quad \cdots \quad + \mathbf{a}_n\, x_n \quad = \quad \mathbf{b}.$$

Slide 12

Span

Definition 5 *Given* p *vectors* $\mathbf{v}_1, \cdots, \mathbf{v}_p$ *in* $\mathbb{R}^n$*, denote by* *Span*$\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ *the set of all linear combinations of* $\mathbf{v}_1, \cdots, \mathbf{v}_p$.

Note:

- Span$\{\mathbf{v}_1, \cdots, \mathbf{v}_p\} \subset \mathbb{R}^n$,
- If $\mathbf{w} \in$ Span$\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$, then there exist numbers $c_1, \cdots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p.$$

Slide 13

Matrices as linear functions

Definition 6 *A linear function* $\mathbf{y} : \mathbb{R}^n \to \mathbb{R}^m$ *is a function* $\mathbf{y}(\mathbf{x})$ *of the form*

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + & \cdots & +a_{1n}x_n + c_1 \\ \vdots & & \vdots \\ a_{m1}x_1 + & \cdots & +a_{mn}x_n + c_m \end{pmatrix},$$

where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \in \mathbb{R}^m, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n,$$

and a_{ij}, c_i *are constants, with* $i = 1, \cdots, m$, *and* $j = 1, \cdots n$.

Slide 14

Introducing the vector $\mathbf{c} \in \mathbb{R}^m$, and the $m \times n$ matrix A as follows,

$$\mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix},$$

then, the linear function $\mathbf{y}(\mathbf{x})$ can be written as,

$$\mathbf{y} = A\mathbf{x} + \mathbf{c}.$$

(Compare it with the expression for a linear function $y : \mathbb{R} \to \mathbb{R}$, that is, $y = ax + c$.)

Slide 15

The product $A\mathbf{x}$ is defined as follows:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + & \cdots & +a_{1n}x_n \\ \vdots & & \vdots \\ a_{m1}x_1 + & \cdots & +a_{mn}x_n \end{pmatrix}.$$

Exercise: Show that this product satisfies the following properties:

- $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$,
- $A(c\mathbf{u}) = cA\mathbf{u}$.

Slide 16

Summary

A system of linear equations

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + & \cdots & +a_{1n}x_n \\ \vdots & & \vdots \\ a_{m1}x_1 + & \cdots & +a_{mn}x_n \end{pmatrix},$$

can be expressed as a linear function, or as a linear combination of the column vectors, respectively,

$$\mathbf{y} = A\mathbf{x}, \quad \mathbf{y} = \mathbf{a}_1 x_1 + \cdots + \mathbf{a}_n x_n,$$

where $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$.

Slide 17

Theorem 1 *Fix and $m \times n$ matrix $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$, and a vector $\mathbf{b} \in \mathbb{R}^m$. Then,*

$$\begin{aligned}
\mathbf{b} \in \textit{Span}\{\mathbf{a}_1, \cdots, \mathbf{a}_n\} \quad &\Leftrightarrow \quad \textit{there exist } x_1, \cdots, x_n, \textit{ such that} \\
&\qquad \mathbf{b} = \mathbf{a}_1 x_1 + \cdots + \mathbf{a}_n x_n, \\
&\Leftrightarrow \quad \mathbf{b} = A\mathbf{x}, \\
&\Leftrightarrow \quad \textit{the echelon form of } [A|\mathbf{b}] \\
&\qquad \textit{has NO row of the form} \\
&\qquad [0 \cdots 0 | b \neq 0].
\end{aligned}$$