Print your name: ____________________ **Sec. Number:** _____
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Math 20C
Quiz 3
November 9, 2005

1. (Sec. 14.6, Probl. 29) Find all points at which the direction of fastest change of the function $f(x,y) = x^2 + y^2 - 2x - 4y$ is given by $\mathbf{i} + \mathbf{j}$.

$$\nabla f(x,y) = \langle 2x-2, 2y-4 \rangle = c(\mathbf{i} + \mathbf{j}) = \langle c, c \rangle,$$

for any $c \in I\!R$. Then, $2x - 2 = c$, and $2y - 4 = c$ which imply $2x - 2 = 2y - 4$, that is,

$$\text{all points in the line } y = x + 1. \tag{1}$$

We consider (1) to be the answer of the problem. (Because that is what appears in the solution part of the book.)

Notice, however, that it is not correct, because it contains one point too much. The right answer is

$$\text{all points in the line } y = x + 1 \text{ except } (1,2).$$

The reason is that $\nabla f(1,2) = \langle 0, 0 \rangle$, as can be seen from the following computation.

$$\begin{aligned} \nabla f(x, x+1) &= \langle 2x-2, 2(x+1)-4 \rangle, \\ &= \langle 2x-2, 2x-2 \rangle, \\ &= (2x-2)\langle 1, 1 \rangle, \\ &= (2x-2)(\mathbf{i} + \mathbf{j}). \end{aligned}$$

2. (Sec. 14.7, Probl. 43) Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid

$$9x^2 + 36y^2 + 4z^2 = 36.$$

Consider the problem in the first octant. Then, $0 < x \leq 2$, $0 < y \leq 1$, and $0 < z \leq 3$. Then, the total volume of the box is 8 times the volume in the first octant, that is, $V(x, y, z) = 8xyz$.

The constraint $9x^2 + 36y^2 + 4z^2 = 36$, implies that

$$2z = \sqrt{36 - 9x^2 - 36y^2}, \quad \Rightarrow z = \frac{3}{2}\sqrt{4 - x^2 - 4y^2}.$$

Then, $V(x, y) = 12\,xy\sqrt{4 - x^2 - 4y^2}$.

$$\begin{aligned} V_x &= 12\,y\left(\sqrt{4 - x^2 - 4y^2} + x\frac{-2x}{2\sqrt{4 - x^2 - 4y^2}}\right), \\ &= \frac{24\,y}{\sqrt{4 - x^2 - 4y^2}}(2 - x^2 - 2y^2). \end{aligned}$$

$$\begin{aligned} V_y &= 12\,x\left(\sqrt{4 - x^2 - 4y^2} + y\frac{-8y}{2\sqrt{4 - x^2 - 4y^2}}\right), \\ &= \frac{12\,x}{\sqrt{4 - x^2 - 4y^2}}(4 - x^2 - 8y^2). \end{aligned}$$

Then, $V_x = 0$, $V_y = 0$ imply $2 - x^2 - 2y^2 = 0$ and $4 - x^2 - 8y^2 = 0$. Therefore, $x^2 = 2 - 2y^2$, and then

$$4 - (2 - 2y^2) - 8y^2 = 0, \quad \Rightarrow 2 - 6y^2 = 0, \quad \Rightarrow y = \frac{1}{\sqrt{3}}.$$

Then,

$$x^2 = 2 - 2\,\frac{2}{3}, \quad \Rightarrow x = \frac{2}{\sqrt{3}},$$

$$z = \frac{3}{2}\sqrt{4 - \frac{4}{3} - 4\frac{1}{3}}, \quad \Rightarrow z = \sqrt{3}.$$

Finally

$$V = 8\frac{2}{\sqrt{3}}\frac{1}{\sqrt{3}}\sqrt{3}, \quad \Rightarrow V = \frac{16}{\sqrt{3}}.$$