Name: ________________________________ **Student Number:** ____________

Math 20C.
Midterm Exam 2
July 23, 2004

Read each question carefully, and answer each question completely.
Show all of your work. No credit will be given for unsupported answers.
Write your solutions clearly and legibly. No credit will be given for illegible solutions.

1. (8 points)
   Consider the function $f(x, y, z) = \sqrt{x + 2yz}$.

   (a) Find the gradient of $f(x, y, z)$.

   $$\nabla f(x, y, z) = \frac{1}{2\sqrt{x + 2yz}} \langle 1, 2z, 2y \rangle.$$

   (b) Find the directional derivative of f at $(0, 2, 1)$ in the direction given by $\langle 0, 3, 4 \rangle$.

   $$\nabla f(0, 2, 1) = \frac{1}{2\sqrt{0+4}} \langle 1, 2, 4 \rangle = \frac{1}{4} \langle 1, 2, 4 \rangle.$$
   $$\mathbf{u} = \frac{1}{\sqrt{9+16}} \langle 0, 3, 4 \rangle = \frac{1}{5} \langle 0, 3, 4 \rangle.$$

   Then,
   $$D_u f(0, 2, 1) = \frac{1}{4} \langle 1, 2, 4 \rangle \cdot \frac{1}{5} \langle 0, 3, 4 \rangle = \frac{1}{20}(6 + 16) = \frac{11}{10}.$$
   Therefore, $D_u f(0, 2, 1) = 11/10$.

   (c) Find the maximum rate of change of f at the point $(0, 2, 1)$.

   $$|\nabla f(0, 2, 1)| = \frac{1}{4} |\langle 1, 2, 4 \rangle| = \frac{1}{4} \sqrt{1 + 4 + 16} = \frac{\sqrt{21}}{4}.$$
   Therefore, the maximum rate of change of f at $(0, 2, 1)$ is $\sqrt{21}/4$.

#	Score
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2. (8 points)

Find any value of the constant a such that the function $f(x,y) = e^{-ax}\cos(y) - e^{-y}\cos(x)$ is solution of Laplace's equation $f_{xx} + f_{yy} = 0$.

$$f_x = -ae^{-ax}\cos(y) + e^{-y}\sin(x), \quad f_y = -e^{-ax}\sin(y) + e^{-y}\cos(x),$$
$$f_{xx} = a^2e^{-ax}\cos(y) + e^{-y}\cos(x), \quad f_{yy} = -e^{-ax}\cos(y) - e^{-y}\cos(x),$$

then

$$\begin{aligned} f_{xx} + f_{yy} &= [a^2e^{-ax}\cos(y) + e^{-y}\cos(x)] + [-e^{-ax}\cos(y) - e^{-y}\cos(x)], \\ &= (a^2 - 1)e^{-ax}\cos(y). \end{aligned}$$

Therefore f is solution of the Laplace equation $f_{xx} + f_{yy} = 0$ if and only if $a = \pm 1$.

3. (8 points)

Let $f(x,y) = 12xy - 2x^3 - 3y^2$.

(a) Find all the critical (stationary) points of f.

$$\nabla f(x,y) = \langle 12y - 6x^2, 12x - 6y\rangle = \langle 0,0\rangle,$$

then,

$$x^2 = 2y, \quad y = 2x, \quad \Rightarrow \quad x(x-4) = 0.$$

Then, there are two solutions, $x = 0$, which implies $y = 0$, and $x = 4$ which implies $y = 8$. That is, there are two critical points, $(0,0)$ and $(4,8)$.

(b) For each critical point of f, determine whether f has a local maximum, local minimum, or saddle point at that point.

$$f_{xx} = -12x, \quad f_{yy} = -6, \quad f_{xy} = 12.$$

$$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = 144\left(\frac{x}{2} - 1\right),$$

Then,

$$D(0,0) = -144 < 0, \quad \Rightarrow \quad (0,0) \text{ is a saddle point of } f.$$

$$D(4,8) = 144(2-1) > 0, \quad f_{xx}(4,8) = (-12)4 < 0, \quad \Rightarrow \quad (4,8) \text{ is a local maximum of } f.$$

4. (8 points)

Use Lagrange multipliers to find the maximum and minimum values of the function $f(x,y) = x^2 + y^2$ subject to the constraint $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$.

Introduce the function $g(x,y) = \frac{1}{4}x^2 + \frac{1}{9}y^2 - 1$. Then, solve

$$\nabla f = \lambda \nabla g, \quad \Rightarrow \quad \langle 2x, 2y \rangle = \lambda \left\langle \frac{1}{2}x, \frac{2}{9}y \right\rangle,$$

that is,

$$2x = \frac{\lambda}{2}x \quad \Rightarrow \quad x(4-\lambda) = 0, \tag{1}$$

$$y = \frac{\lambda}{9}y \quad \Rightarrow \quad y(9-\lambda) = 0. \tag{2}$$

Therefore, there are 4 possibilities:

- $x = 0$, then the constraint implies $y = \pm 3$, so the points are $(0, \pm 3)$.
- $\lambda = 4$, then Eq. (2) implies $y = 0$, then the constraint says that $x = \pm 2$, so the points are $(\pm 2, 0)$.
- $y = 0$, and then we recover points $(\pm 2, 0)$.
- $\lambda = 9$ and then Eq. (1) says $x = 0$ and we recover points $(0, \pm 3)$.

Summarizing, we have four critical points, $(\pm 2, 0)$ and $(0, \pm 3)$. Now, $f(\pm 2, 0) = 4$, and $f(0, \pm 3) = 9$, so the former two points are minimums, and the latter two are maximums.