Print Name: ______________________ **Student Number:** ______________

Section Time: ______________

Math 20C.
Midterm Exam 2
November 21, 2005

Read each question carefully, and answer each question completely.
Show all of your work. No credit will be given for unsupported answers.
Write your solutions clearly and legibly. No credit will be given for illegible solutions.

1. (6 points)

   (a) Find the tangent plane approximation $L(x, y)$ of the function
   $$f(x, y) = \sin(2x + 3y) + 1$$
   at the point $(-3, 2)$.

   (b) Use the approximation above to estimate the value of $f(-2.8, 2.3)$.

(a)
$$f_x(x, y) = 2\cos(2x + 3y), \quad f_y(x, y) = 3\cos(2x + 3y),$$
then
$$\begin{aligned} f_x(-3, 2) &= 2\cos(-6 + 6) = 2, \\ f_y(-3, 2) &= 3\cos(-6 + 6) = 3, \\ f(-3, 2) &= \sin(-6 + 6) + 1 = 1. \end{aligned}$$
Then, the linear approximation
$$L(x, y) = f_x(-3, 2)\,(x + 3) + f_y(-3, 2)\,(y - 2) + f(-3, 2)$$
is given by,
$$L(x, y) = 2(x + 3) + 3(y - 2) + 1.$$

(b) The linear approximation of $f(-2.8, 2.3)$ is $L(-2.8, 2.3)$, and
$$L(-2.8, 2.3) = 2(-2.8 + 3) + 3(2.3 - 2) + 2 = 2(0.2) + 3(0.3) + 1 = 2.3,$$
then, the result is $L(-2.8, 2.3) = 2.3$.

#	Score
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2. (6 points) Find the absolute maximum and absolute minimum of

$$f(x,y) = 2 + xy - 2x - \frac{1}{4}y^2$$

in the closed triangular region with vertices given by $(0,0)$, $(1,0)$, and $(0,2)$. Justify your answer.

We start finding the critical points inside the triangular region.

$$\nabla f(x,y) = \left\langle y - 2, x - \frac{1}{2}y \right\rangle = \langle 0, 0\rangle, \quad \Rightarrow \quad y = 2, \quad y = 2x.$$

The solution is $(1,2)$. This point is outside in the triangular region given by the problem, so there is no critical point inside the region.

We now find the candidates for absolute maximum and minimum on the borders of the triangular region.

- $y = 0$, $x \in [0,1]$. Then,

  $$f(x,0) = 2 - 2x, \quad \Rightarrow f_x(x,0) = -2 \neq 0.$$

  Then, the candidate is $(0,0)$, and $f(0,0) = 2$.
  The other candidate is $(1,0)$, and $f(1,0) = 0$.
- $x = 0$, $y \in [0,2]$. Then,

  $$f(0,y) = 2 - \frac{1}{4}y^2, \quad \Rightarrow f_y(0,y) = -\frac{1}{2}y = 0, \quad \Rightarrow \quad y = 0,$$

  then we recover $(0,0)$.
  The other candidate is $(0,2)$, and $f(0,2) = 1$.
- $y = 2 - 2x$, $x \in [0,1]$. Then,

  $$\begin{aligned} f(x, 2-2x) &= 2 + x(2-2x) - 2x - \frac{1}{4}(2-2x)^2, \\ &= 2 + 2x - 2x^2 - 2x - (x^2 - 2x + 1), \\ &= 1 + 2x - 3x^2, \end{aligned}$$

  therefore,

  $$f_x(x, 2-2x) = 2 - 6x = 0, \quad \Rightarrow \quad x = \frac{1}{3}, \quad \Rightarrow \quad y = \frac{4}{3},$$

  so the candidate is $(1/3, 4/3)$, and one has

  $$f(1/3, 4/3) = 2 + \frac{4}{9} - \frac{2}{3} - \frac{1}{4}\frac{16}{9} = \frac{4}{3},$$

  that is $f(1/3, 4/3) = 4/3$.

Therefore, the absolute maximum is at $(0,0)$, and the absolute minimum is at $(1,0)$.

3. (6 points) Using Lagrange multipliers find the maximum and minimum values of

$$f(x, y) = 2(x + 1)y,$$

subject to the constraint

$$x^2 + y^2 = 1.$$

Show all your work.

Let $g(x, y) = x^2 + y^2 - 1$, then

$$\nabla f = \lambda \nabla g \quad \Rightarrow \quad 2\langle y, x + 1\rangle = 2\lambda\langle x, y\rangle,$$

that is

$$y = \lambda x, \quad x + 1 = \lambda y, \quad \Rightarrow \quad x + 1 = \lambda^2 x,$$

then

$$x(\lambda^2 - 1) = 1, \quad \Rightarrow x = \frac{1}{\lambda^2 - 1}, \quad \Rightarrow \quad y = \frac{\lambda}{\lambda^2 - 1}.$$

Then, the constraint says that

$$\frac{1}{(\lambda^2 - 1)^2} + \frac{\lambda^2}{(\lambda^2 - 1)^2} = 1, \quad \Rightarrow \quad (\lambda^2 - 1)^2 = \lambda^2 + 1,$$

then

$$\lambda^4 - 2\lambda^2 + 1 = \lambda^2 + 1, \quad \Rightarrow \quad \lambda^2(\lambda^2 - 3) = 0,$$

which implies that $\lambda = 0$ or $\lambda = \pm\sqrt{3}$.

In the first case one gets $x = -1$ and $y = 0$, that is, the point $(-1, 0)$. Then, $f(-1, 0) = 0$.

In the second case one gets $x = 1/2$, and $y = \pm\sqrt{3}/2$, that is, the points $(1/2, \pm\sqrt{3}/2)$. Then, $f(1/2, \pm\sqrt{3}/2) = \pm 3\sqrt{3}/2$.

Therefore, the maximum is at $(1/2, \sqrt{3}/2)$, and the minimum is at $(1/2, -\sqrt{3}/2)$.

4. (6 points) Compute the double integral of the function

$$f(x, y) = \frac{x}{y} e^{3x^2},$$

in the domain

$$R = \{(x, y) \in \mathbb{R}^2 : \quad 0 \leq x \leq 1, \quad 1 \leq y \leq 3\}.$$

Show all your work.

$$\begin{aligned}
\int\int_R f \, dA &= \int_1^3 \int_0^1 \frac{x}{y} e^{3x^2} \, dx \, dy, \\
&= \left(\int_1^3 \frac{1}{y} dy \right) \left(\int_0^1 e^{3x^2} x dx \right), \\
&= \left(\ln(y)|_1^3 \right) \left(\int_0^3 \frac{1}{6} e^u du \right), \\
&= \frac{1}{6} \ln(3)(e^3 - 1),
\end{aligned}$$

where we did the substitution $u = 3x^2$, $du = 6x \, dx$ to obtain the third line above.