Print Name: ______________________ **Student Number:** ______________

Section Time: ______________

Math 20C.
Final Exam
December 8, 2005

Read each question carefully, and answer each question completely.
Show all of your work. No credit will be given for unsupported answers.
Write your solutions clearly and legibly. No credit will be given for illegible solutions.

1. (8 Pts.) Find the equation of the plane that contains both the point $(1, 0, -1)$ and the line $x = t$, $y = -1 + 2t$, $z = 3t$.

The plane is determined by a point in the plane and the normal vector. A point in the plane is $P_0 = (1, 0, -1)$.

To compute the normal vector $\mathbf{n}$, notice that the equation of the line is given by $\mathbf{r}(t) = \langle 0, -1, 0\rangle + \langle 1, 2, 3\rangle t$. Denote $\mathbf{v} = \langle 1, 2, 3\rangle$, and $P_1 = \mathbf{r}(0) = (0, -1, 0)$. Then, $\vec{P_1P_0} = \langle 1, 1, -1, \rangle$. Therefore,

$$\mathbf{n} = \mathbf{v} \times \vec{P_1P_0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 1 & 1 & -1 \end{vmatrix} = \langle (-2-3), -(-1-3), (1-2)\rangle,$$

$$\mathbf{n} = \langle -5, 4, -1\rangle.$$

Then, the equation of the plane is

$$-5(x-1) + 4(y-0) - (z+1) = 0, \quad \Rightarrow \quad -5x + 5 + 4y - z - 1 = 0,$$

$$5x - 4y + z = 4.$$

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2. (8 Pts.) Find the values of the constants b and c such that the function

$$f(t,x) = \sin(x - bt) + \cos(cx + t)$$

is solution of the wave equation $f_{tt} = 9 f_{xx}$.

$$\begin{aligned} f_t &= -b\cos(x-bt) - \sin(cx+t), \\ f_{tt} &= -b^2 \sin(x-bt) - \cos(cx+t), \\ f_x &= \cos(x-bt) - c\sin(cx+t), \\ f_{xx} &= -\sin(x-bt) - c^2\cos(cx+t), \end{aligned}$$

therefore,

$$0 = f_{tt} - 9f_{xx} = [-b^2\sin(x-bt) - \cos(cx+t)] - 9[-\sin(x-bt) - c^2\cos(cx+t)],$$

$$0 = (9 - b^2)\sin(x-bt) + (9c^2 - 1)\cos(cx+t) \quad \Rightarrow \quad b = \pm 3, \quad c = \pm\frac{1}{3}.$$

3. (6 Pts.) Consider the function $z(t) = f(x(t), y(t))$, where

$$f(x,y) = (x^2 + 2y)^{1/2}, \quad x(t) = e^{3t}, \quad y(t) = e^{-3t}.$$

Compute $\dfrac{d}{dt} z(t)$.

$$\begin{aligned}
\frac{dz}{dt} &= f_x \frac{dx}{dt} + f_y \frac{dy}{dt}, \\
&= \frac{1}{2\sqrt{x^2 + 2y}} 2x \frac{dx}{dt} + \frac{1}{2\sqrt{x^2 + 2y}} 2 \frac{dy}{dt}, \\
&= \frac{1}{\sqrt{e^{6t} + 2e^{-3t}}} e^{3t} 3e^{3t} + \frac{1}{\sqrt{e^{6t} + 2e^{-3t}}} (-3) e^{-3t}, \\
&= 3 \frac{(e^{6t} - e^{-3t})}{\sqrt{e^{6t} + 2e^{-3t}}}.
\end{aligned}$$

4. (8 Pts.) The function $z(x, y)$ is defined implicitly by the equation $z^2xy = \sin(2y + z)$. Compute the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ as functions of x, y and z.

$$z^2y + xy2zz_x = \cos(2y + z)z_x, \Rightarrow z^2y + [2xyz - \cos(2y + z)]z_x = 0,$$
$$z_x = \frac{yz^2}{[\cos(2y + z) - 2xyz]}.$$

$$z^2x + xy2zz_y = \cos(2y + z)(2 + z_y), \Rightarrow z^2x + [2xyz - \cos(2y + z)]z_y = 2\cos(2y + z),$$
$$z_y = -\frac{2\cos(2y + z) - xz^2}{[2xyz - \cos(2y + z)]}.$$

5. (10 Pts.) Reparametrize the curve $\mathbf{r}(t) = (e^{3t}\cos(3t), 1, e^{3t}\sin(3t))$, with respect to the arc length measured from the point where $t = 0$ in the direction of increasing t.

$$\begin{aligned}\mathbf{v}(t) &= \langle 3e^{3t}\cos(3t) - 3e^{3t}\sin(3t), 0, 3e^{3t}\sin(3t) + 3e^{3t}\cos(3t)\rangle, \\ &= 3e^{3t}\langle[\cos(3t) - \sin(3t)], 0, [\sin(3t) + \cos(3t)]\rangle.\end{aligned}$$

$$\begin{aligned}|\mathbf{v}(t)| &= 3e^{3t}\sqrt{[\cos(3t) - \sin(3t)]^2 + [\sin(3t) + \cos(3t)]^2}, \\ &= 3e^{3t}\sqrt{\cos^2(3t) + \sin^2(3t) - 2\sin(3t)\cos(3t) + \sin^2(3t) + \cos^2(3t) + 2\sin(3t)\cos(3t)}, \\ &= 3e^{3t}\sqrt{2\cos^2(3t) + 2\sin^2(3t)}, \\ &= 3\sqrt{2}e^{3t}.\end{aligned}$$

Then,

$$s(t) = 3\sqrt{2}\int_0^t e^{3u}du = \sqrt{2}(e^{3t} - 1),$$

so $e^{3t} = 1 + s/\sqrt{2}$, and $3t = \ln(1 + s/\sqrt{2})$. Then,

$$\mathbf{r}(t) = \left\langle \left[1 + \frac{s}{\sqrt{2}}\right]\cos\left(1 + \frac{s}{\sqrt{2}}\right), 1, \left[1 + \frac{s}{\sqrt{2}}\right]\sin\left(1 + \frac{s}{\sqrt{2}}\right)\right\rangle.$$

6. (10 Pts.) Consider the function $f(x, y) = x^3 - 6xy + 8y^3$.

(a) Find the critical points of f.

(b) For each critical point determine whether it is a local maximum, local minimum, or a saddle point.

(a)

$$\nabla f = \langle 3x^2 - 6y, -6x + 24y^2 \rangle = \langle 0, 0 \rangle,$$

$$x^2 = 2y, \quad 4y^2 = x, \quad \Rightarrow \quad 4\frac{x^4}{4} = x,$$

$$x(x^3 - 1) = 0.$$

If $x = 0$, then $y = 0$, so we have the point $(0, 0)$.

If $x = 1$, then $y = 1/2$, so we have the point $(1, 1/2)$.

(b)

$$f_{xx} = 6x, \quad f_{yy} = 48y, \quad f_{xy} - 6.$$

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 36(8xy - 1).$$

Then, $D(0, 0) = -36 < 0 \Rightarrow (0, 0)$ is a saddle point of f.

Then, $D(1, 1/2) = 36(4 - 1) > 0$ and $f_{xx}(1, 1/2) = 6 > 0$, then $(1, 1/2)$ is a local minimum.

7. (10 Pts.) Find the maximum and minimum values of the function $f(x,y) = 2x^2+y^2-2y$ subject to the constraint $x^2+y^2=4$.

Let $g(x,y) = x^2+y^2-4$.

$$\nabla f = \langle 4x, 2y-2\rangle, \quad \nabla g = \langle 2x, 2y\rangle,$$

then the equation $\nabla f = \lambda \nabla g$ implies

$$2x = \lambda x, \quad y-1 = \lambda y.$$

The first equation implies $x(\lambda-2)=0$. Then, $\lambda = 2$ or $x=0$.

If $\lambda = 2$, then $y-1 = 2y$, that is, $y=-1$, and then $x^2 = 4-1$ so $x = \pm\sqrt{3}$. So we have the points $(\pm\sqrt{3}, -1)$.

If $x=0$, then $y=\pm 2$, so we have the points $(0,\pm 2)$.

Now,

$$f(\pm\sqrt{3},-1) = 2(3)+1+2 = 9, \quad f(0,\pm 2) = 4 \mp 4, \quad \Rightarrow \quad f(0,2) = 0, \quad f(0,-2) = 8.$$

Therefore, $(\pm\sqrt{3},-1)$ are maximum of f, and $(0,2)$ is a minimum of f.

8. (a) (10 Pts.) Sketch the region of integration, D, whose area is given by the double integral

$$\int\int_D dA = \int_0^3 \int_{\frac{2}{3}x}^{2\sqrt{x/3}} dy\, dx.$$

(b) Compute the double integral given in (a).

(c) Change the order of integration in the integral given in (a). (You don't need to compute the integral again.)

(b)

$$\begin{aligned}
\int\int_D dA &= \int_0^3 \int_{\frac{2}{3}x}^{2\sqrt{x/3}} dy\, dx, \\
&= \int_0^3 \left[\frac{2}{\sqrt{3}} x^{1/2} - \frac{2}{3}x \right] dx, \\
&= \frac{2}{\sqrt{3}} \frac{2}{3} \left(x^{3/2}\Big|_0^3 \right) - \frac{1}{3} \left(x^2\Big|_0^3 \right), \\
&= \frac{4}{3\sqrt{3}} (\sqrt{3})^3 - \frac{1}{3} 9, \\
&= 4 - 3, \\
&= 1.
\end{aligned}$$

(c)

$$\int\int_D dA = \int_0^2 \int_{\frac{3}{4}y^2}^{\frac{3}{2}y} dx\, dy.$$

9. (10 Pts.) Compute the integral

$$I = \int_0^1 \int_0^x \int_0^{4-x^2} xz\, dy\, dz\, dx$$

$$\begin{aligned}
I &= \int_0^1 \int_0^x \int_0^{4-x^2} xz\, dy\, dz\, dx, \\
&= \int_0^1 \int_0^x x(4-x^2) z dz\, dx, \\
&= \int_0^1 x(4-x^2)\frac{1}{2}x^2\, dx, \\
&= \frac{1}{2}\int_0^1 (4x^3 - x^5)\, dx, \\
&= \frac{1}{2}\left[x^4|_0^1 - \frac{1}{6}x^6|_0^1 \right], \\
&= \frac{1}{2}\left(1 - \frac{1}{6}\right), \\
&= \frac{5}{12}.
\end{aligned}$$

10. (10 Pts.) Consider the region of $D \subset I\!R^3$ given by

$$D = \{(x, y, z) \in I\!R^3 : x^2 + y^2 \leq 1, \quad 0 \leq z \leq 1 + x^2 + y^2\}.$$

(a) Sketch the region D.

(b) Compute the volume of that region.

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^1 \int_0^{1+r^2} dz \, r dr \, d\theta, \\ &= 2\pi \int_0^1 (1 + r^2) r dr, \\ &= 2\pi \int_1^2 u \frac{1}{2} du, \\ &= 2\pi \frac{1}{4} \left(u^2 \Big|_1^2 \right), \\ &= \frac{3}{2}\pi. \end{aligned}$$