Name: ____________________
(Use capitals)
Student number: ____________________

Math 20C
Final Exam
July 31, 2004

Read each question carefully, and answer each question completely.
Show all of your work. No credit will be given for unsupported answers.
Write your solutions clearly and legibly. No credit will be given for illegible solutions.

1. (8 points)

   (a) Find the angle between the planes $2x + y + 3z = 1$ and $-x - 3y + 2z = 5$.

$$\vec{n}_1 = \langle 2, 1, 3 \rangle, \quad \vec{n}_2 = \langle -1, -3, 2 \rangle,$$
$$|\vec{n}_1| = \sqrt{4+1+9} = \sqrt{14}, \quad |\vec{n}_2| = \sqrt{4+1+9} = \sqrt{14}.$$
$$\cos(\theta) = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} = \frac{-2-3+6}{14} = \frac{1}{14}.$$

   So the answer is $\cos(\theta) = \dfrac{1}{14}$.

   (b) Find a vector parallel to the line of intersection of the planes given in (a).

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = \langle (2+9), -(4+3), (-6+1) \rangle,$$
$$\vec{v} = \langle 11, -7, -5 \rangle.$$

#	Score
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2. (8 points)

A particle has the velocity function $\vec{v}(t) = \langle -3\sin(t), 4, 3\cos(t)\rangle$.

(a) Find the particle acceleration $\vec{a}(t)$.

$$\vec{a}(t) = \vec{v}(t)' = \langle -3\cos(t), 0, -3\sin(t)\rangle.$$

(b) The particle initial position is given by $\vec{r}(0) = \langle 3, 1, 0\rangle$. Find the particle position function $\vec{r}(t)$.

$$\vec{r}(t) = \langle 3\cos(t) + x_0, 4t + y_0, 3\sin(t) + z_0\rangle,$$

$$\vec{r}(0) = \langle 3 + x_0, y_0, z_0\rangle = \langle 3, 1, 0\rangle,$$

$$x_0 = 0, \quad y_0 = 1, \quad z_0 = 0.$$

Then, the answer is

$$\vec{r}(t) = \langle 3\cos(t), 4t + 1, 3\sin(t)\rangle.$$

(c) Reparametrize the curve $\vec{r}(t)$ with respect to the arc length measured from the point where $t = 0$, in the direction of increasing t.

$$|\vec{v}(t)| = \sqrt{9\cos^2(t) + 16 + 9\sin^2(t)} = \sqrt{9 + 16} = 5,$$

$$s(t) = \int_0^t 5\,du = 5t, \quad \Rightarrow \quad t = \frac{s}{5}.$$

Then, the answer is

$$\vec{r}(s) = \left\langle 3\cos\left(\frac{s}{5}\right), \frac{4}{5}s + 1, 3\sin\left(\frac{s}{5}\right)\right\rangle.$$

3. (8 points)

(a) Find an equation for the plane tangent to the graph of $f(x,y) = \frac{1}{\pi}\cos\left(\frac{\pi}{2}x^2y\right)$ at the point $(-1,1)$.

$$z = L_{(-1,1)}(x,y) = f_x(-1,1)\,(x+1) + f_y(-1,1)\,(y-1) + f(-1,1),$$

$$f_x = -\frac{1}{\pi}\sin\left(\frac{\pi}{2}x^2y\right)\frac{\pi}{2}2xy, \quad \Rightarrow \quad f_x = -xy\sin\left(\frac{\pi}{2}x^2y\right).$$

$$f_y = -\frac{1}{\pi}\sin\left(\frac{\pi}{2}x^2y\right)\frac{\pi}{2}x^2, \quad \Rightarrow \quad f_x = -\frac{x^2}{2}\sin\left(\frac{\pi}{2}x^2y\right).$$

then,

$$f_x(-1,1) = 1, \quad f_y(-1,1) = -\frac{1}{2}, \quad f(-1,1) = 0.$$

Then, the solution is

$$z = (x+1) - \frac{1}{2}(y-1).$$

(b) Find the linear approximation for $f(-1.1, 1.2)$.

$$L(-1.1, 1.2) = (-1.1+1) - \frac{1}{2}(1.2-1) = -0.1 - 0.1 = -0.2,$$

so the answer is $L(-1.1, 1.2) = -0.2$.

4. (8 points)

Consider the function $f(x, y, z) = 2\sin(xyz)$.

(a) Find a unit vector in the direction of greatest increase of f at the point $(\pi, 1, 1)$.

$$\nabla f(x, yz) = 2\cos(xyz)\langle yz, xz, xy\rangle,$$

$$\nabla f(\pi, 1, 1) = -2\langle 1, \pi, \pi\rangle, \quad \Rightarrow \quad |\nabla f(\pi, 1, 1)| = 2\sqrt{1+\pi^2}.$$

The answer is then,

$$\vec{u} = -\frac{1}{\sqrt{1+2\pi^2}}\langle 1, \pi, \pi\rangle.$$

(b) Find the directional derivative of f at the point $(\pi, 1, 1)$ in the direction given by the vector $\langle 1, 1, -1\rangle$.

$$\vec{v} = \langle 1, 1, -1\rangle, \quad \Rightarrow \quad |\vec{v}| = \sqrt{3}.$$

$$\vec{u} = \frac{1}{\sqrt{3}}\langle 1, 1, -1\rangle.$$

$$D_u f = \nabla f(\pi, 1, 1) \cdot \frac{1}{\sqrt{3}}\langle 1, 1, -1\rangle = -\frac{2}{\sqrt{3}}(1 + \pi - \pi) = -\frac{2}{\sqrt{3}}.$$

So the answer is

$$D_u f = -\frac{2}{\sqrt{3}}.$$

5. (8 points)

Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = xy$ subject to the constraint $2x + y^2 = 3$.

Let $g(x, y) = 2x + y^2 - 3$. Then,

$$\nabla f = \langle y, x\rangle, \quad \nabla g = \langle 2, 2y\rangle,$$

then the equation $\nabla f = \lambda \nabla g$ implies

$$y = 2\lambda, \quad x = 2\lambda y, \quad \Rightarrow \quad x = y^2,$$

and the constraint gives $2x + x = 3$ that is $x = 1$, so, $y = \pm 1$. Then, the critical points are $(1, \pm 1)$. Notice that

$$f(1, \pm 1) = \pm 1,$$

so $(1, 1)$ is a maximum, and $(, -1)$ is a minimum.

6. (8 points)

(a) Sketch the region of integration, D, whose area is given by the double integral $\int\int_D dA = \int_0^2 \int_{\frac{3}{2}x}^3 dy\, dx$.

The triangle with vertices $(0,0)$, $(0,3)$ and $(2,3)$.

(b) Compute the double integral given in (a).

$$\begin{aligned}\int\int_D dA &= \int_0^2 \int_{\frac{3}{2}x}^3 dy\, dx, \\ &= \int_0^2 \left(3 - \frac{3}{2}x\right) dx, \\ &= 3\left[x\Big|_0^2 - \frac{1}{4}\, x^2\Big|_0^2 \right], \\ &= 3(2-1), \\ &= 3.\end{aligned}$$

(c) Change the order of integration in the integral given in (a). (You don't need to compute the integral again.)

$$\int\int_d dA = \int_0^3 \int_0^{\frac{2}{3}y} dx\, dy.$$

7. (8 points)

Use polar coordinates to compute the double integral of $f(x,y) = xy$ in the region $D = \{(x,y) : \quad 0 \leq x, \quad x^2 + y^2 \leq 4\}$.

$$\begin{aligned}
\int\int_D xy\, dA &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 r^2 \cos(\theta)\sin(\theta)\, r dr\, d\theta, \\
&= \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)\sin(\theta)\, d\theta\right] \left[\int_0^2 r^3\, dr\right], \\
&= \frac{1}{2}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(2\theta)\, d\theta\right] \left[\frac{1}{4}\, r^4\Big|_0^2\right], \\
&= -\frac{1}{4}\left[\cos(2\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right] 4, \\
&= 0.
\end{aligned}$$

8. (8 points)

Use a triple integral to compute the volume of the tetrahedron whose sides are given by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + 2y + z = 2$.

$$
\begin{aligned}
V &= \int_0^1 \int_0^{1-x} \int_0^{2-2x-2y} dz\, dy\, dx, \\
&= 2\int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx, \\
&= 2\int_0^1 \left[(1-x)\left(y|_0^{1-x}\right) - \frac{1}{2}\left(y^2|_0^{1-x}\right)\right] dx, \\
&= 2\int_0^1 \left[(1-x)^2 - \frac{1}{2}(1-x)^2\right] dx, \\
&= \int_0^1 (1-x)^2 dx, \\
&= \int_1^0 u^2(-du), \\
&= \int_0^1 u^2 du, \\
&= \frac{1}{3}.
\end{aligned}
$$