The problem of initial data in general relativity.

Gabriel Nagy

October 26, 2010.
Geometric Analysis Seminar, MSU.

The Cauchy problem for Einstein's equations of classical general relativity contains a difficulty usually not present in other evolution equations. The initial data fields cannot be freely specified but they must satisfy certain constraint equations. In this sense the Cauchy problem for Einstein's equations is similar to Maxwell's equations of electromagnetism. We review a well known method, called the conformal method, to find solutions of the Einstein constraint equations. We summarize few old and new results on the existence of solutions to the constraint equations representing closed manifolds with constant or non-constant mean curvature.

The problem of initial data in general relativity.

- A constrained Cauchy problem: Maxwell's equations.
- The problem of initial data for Maxwell's equations.

- The problem of initial data for Einstein's equations.
- The conformal method to solve the constraint equations.

- Results on closed manifolds:
  - The CMC case. (Understood by 1995.)
  - The near-CMC case. (First result in 1996.)
  - The far-CMC case. (First results in 2008.)

A constrained Cauchy problem: Maxwell's equations.

Definition

Consider a manifold $(\mathbb{R}^4, \eta)$, with $\eta = \text{diag}[-1, 1, 1, 1]$ the Minkowski metric, and fix on $\mathbb{R}^4$ a one-form field $\mathbf{J} \in \Lambda(\mathbb{R}^4)$, satisfying the conservation equation $d(*\mathbf{J}) = 0$, with $*$ the Hodge star operator. The two-form field $\mathbf{F} \in \Lambda^2(\mathbb{R}^4)$ is the *electromagnetic field* on $\mathbb{R}^4$ produced by the current $\mathbf{J}$ iff the two-form $\mathbf{F}$ is solution of the *Maxwell equations*

$$d(*\mathbf{F}) = 4\pi(*\mathbf{J}), \qquad d\mathbf{F} = 0.$$

Remark: To formulate a Cauchy problem requires to rewrite Maxwell's equations as evolution equations in spacetime.

Notation: Let $\delta = \text{diag}[1, 1, 1]$ be the Euclidean metric on spacelike hypersurfaces on $\mathbb{R}^4$.

A constrained Cauchy problem: Maxwell's equations.

Theorem (Space and time decomposition)

Consider Maxwell's equations

$$d(*\mathbf{F}) = 4\pi(*\mathbf{J}), \qquad d\mathbf{F} = 0.$$

Given a foliation $\{\Sigma_t\}$ of $(\mathbb{R}^4, \eta)$ by spacelike hypersurfaces with unit, future directed, normal vector field $\mathbf{n}$, introduce the electric and magnetic vector fields, respectively,

$$\mathbf{E} = -\eta^{-1}\big(i_{\mathbf{n}}\mathbf{F}\big), \qquad \mathbf{B} = \eta^{-1}\big(i_n(*\mathbf{F})\big),$$

and the source fields $\rho = i_{\mathbf{n}}\mathbf{J}$ and $\mathbf{j} = -\delta^{-1}(\mathbf{J})$. Then, Maxwell's equations are given by (cgs units, $c = 1$)

$$\partial_t\mathbf{E} - \nabla\wedge\mathbf{B} = -4\pi\mathbf{j}, \quad \partial_t\mathbf{B} + \nabla\wedge\mathbf{E} = \mathbf{0}, \quad \textit{(Evolution)},$$
$$\nabla\cdot\mathbf{E} = 4\pi\rho, \qquad \nabla\cdot\mathbf{B} = 0, \quad \textit{(Constraint)},$$

where the sources satisfy $\partial_t\rho + \nabla\cdot\mathbf{j} = 0$.

A constrained Cauchy problem: Maxwell's equations.

Theorem (Evolution equations)

Given smooth source fields ρ, $\mathbf{j}$ *on* $\mathbb{R}^4$ *and* $\mathbf{E}_0$, $\mathbf{B}_0$ *on* Σ_0, *there exists a unique smooth solution to the Cauchy problem*

$$\partial_t \mathbf{E} - \nabla \wedge \mathbf{B} = -4\pi \mathbf{j}\,, \qquad (1)$$

$$\partial_t \mathbf{B} + \nabla \wedge \mathbf{E} = \mathbf{0}. \qquad (2)$$

satisfying the initial conditions $\mathbf{E}\Big|_{\Sigma_0} = \mathbf{E}_0$ *and* $\mathbf{B}\big|_{\Sigma_0} = \mathbf{B}_0$.

Theorem (Constraint propagation)

Given any smooth solution of Eqs. (1)-(2), the constraint fields

$$C_E = \nabla \cdot \mathbf{E} - 4\pi\rho, \qquad C_B = \nabla \cdot \mathbf{B},$$

satisfy the equations $\quad \partial_t C_E = 0, \qquad \partial_t C_B = 0.$

Proof: $\nabla \cdot (\nabla \wedge \mathbf{v}) = 0$, $\partial_t \rho + \nabla \cdot \mathbf{j} = 0$. $(dd = 0,\ d(*\mathbf{J}) = 0.)$ □

The problem of initial data in general relativity.

- A constrained Cauchy problem: Maxwell's equations.
- **The problem of initial data for Maxwell's equations.**
- The problem of initial data for Einstein's equations.
- The conformal method to solve the constraint equations.
- Results on closed manifolds:
  - The CMC case.
  - The near-CMC case.
  - The far-CMC case.

The problem of initial data for Maxwell's equations.

Definition

The 3-surface and the fields $(\mathbb{R}^3, \mathbf{E}_0, \mathbf{B}_0, \rho)$ form an *initial data set for Maxwell's equations* iff hold

$$\nabla \cdot \mathbf{E}_0 = 4\pi\rho, \qquad \nabla \cdot \mathbf{B}_0 = 0. \tag{3}$$

Remarks:

- The symbol in Eqs. (3) is surjective but not injective.
- Transform the Eqs. (3) into PDEs with invertible symbol.

Theorem (Grad-Curl decomposition)

Fix arbitrary vector fields $\mathbf{A}_E$, $\mathbf{A}_B$, *and denote by* ϕ_E, ϕ_B *scalar fields on* $(\mathbb{R}^3, \delta)$. *The vector fields* $\mathbf{E}_0 = \nabla\phi_E + \nabla \wedge \mathbf{A}_E$ *and* $\mathbf{B}_0 = \nabla\phi_B + \nabla \wedge \mathbf{A}_B$ *are solutions of the Maxwell constraint equations iff the fields* ϕ_E *and* ϕ_B *are solutions of*

$$\Delta\phi_E = 4\pi\rho, \qquad \Delta\phi_B = 0, \qquad \phi_E, \phi_B \to 0 \text{ as } |x| \to \infty.$$

The problem of initial data in general relativity.

- A constrained Cauchy problem: Maxwell's equations.
- The problem of initial data for Maxwell's equations.
- **The problem of initial data for Einstein's equations.**
- The conformal method to solve the constraint equations.
- Results on closed manifolds:
  - The CMC case.
  - The near-CMC case.
  - The far-CMC case.

The problem of initial data for Einstein's equations.

Definition

The manifold and fields $(\mathcal{M}, \hat{h}_{ab}, \hat{k}^{ab}, \hat{j}^a, \hat{\rho})$ form an *initial data set for Einstein's equation* iff hold:

(a) $\mathcal{M}$ is a 3-dimensional smooth manifold;

(b) $\hat{h}_{ab}$ is a Riemannian metric on $\mathcal{M}$;

(c) $\hat{k}^{ab}$ is a symmetric tensor field on $\mathcal{M}$;

(d) $\hat{j}^a$ and $\hat{\rho}$ are a vector and a non-negative scalar field on $\mathcal{M}$;

and the following equations hold on $\mathcal{M}$,

$$\hat{R} + \hat{k}^2 - \hat{k}_{ab}\hat{k}^{ab} = 2\kappa\hat{\rho}, \qquad \hat{\nabla}_a \hat{k}^{ab} - \hat{\nabla}^b \hat{k} = \kappa \hat{j}^b, \tag{4}$$

together with the energy condition $-\hat{\rho}^2 + \hat{j}_a\hat{j}^a \leqslant 0$, with strict inequality at points in $\mathcal{M}$ where $\hat{\rho} \neq 0$. Here $\hat{\nabla}_a$ is the Levi-Civita connection of $\hat{h}_{ab}$, $\hat{R}$ is the Ricci scalar of $\hat{\nabla}_a$, and $k = 8\pi$.

The problem of initial data for Einstein's equations.

Remarks:

Once the spacetime is constructed with the initial data above, the following statements hold:

- The 3-dim manifold $\mathcal{M}$ is space at the initial time;
- The fields $\hat{h}_{ab}$ and $\hat{k}^{ab}$ are the first and second fundamental forms of $\mathcal{M}$ when embedded in the spacetime;
- The $\hat{\mathbf{j}}^a$ and $\hat{\rho}$ are the matter and radiation at the initial time.
- The Eqs. $\hat{R} + \hat{k}^2 - \hat{k}_{ab}\hat{k}^{ab} = 2\kappa\hat{\rho}$, and $\hat{\nabla}_a \hat{k}^{ab} - \hat{\nabla}^b \hat{k} = \kappa \hat{j}^b$, are the Gauss and Codazzi equations written in terms of intrinsic fields of the 3-surface $\mathcal{M}$ after using Einstein's equations.
- The constraint eqs. have surjective but not injective symbol.
- The energy condition $-\hat{\rho}^2 + \hat{j}_a\hat{j}^a \leqslant 0$, on the matter fields is the reason why the constraint equations are indeed equations for $\hat{h}_{ab}$ and $\hat{k}^{ab}$.

The problem of initial data in general relativity.

- A constrained Cauchy problem: Maxwell's equations.
- The problem of initial data for Maxwell's equations.
- The problem of initial data for Einstein's equations.
- **The conformal method to solve the constraint equations.**
- Results on closed manifolds:
  - The CMC case.
  - The near-CMC case.
  - The far-CMC case.

Theorem (Conformal decomposition)

Fix on a 3-dimensional manifold $\mathcal{M}$ the following:

(a) *A Riemannian metric h_{ab} with conn. ∇_a and Ricci sclr. R;*

(b) *A symmetric tensor σ^{ab} with $h_{ab}\sigma^{ab} = 0$ and $\nabla_a \sigma^{ab} = 0$;*

(c) *Scalar fields τ and ρ and a vector field $\mathbf{j}^{\,a}$, satisfying the condition $-\rho^2 + h_{ab}\,\mathbf{j}^{\,a}\mathbf{j}^{\,b} \leqslant 0$, with $<$ where $\rho \neq 0$.*

Let $(\mathcal{L}\mathbf{v})^{ab} = \nabla^a \mathbf{v}^b + \nabla^b \mathbf{v}^a - (2/3)(\nabla_c \mathbf{v}^c)\, h^{ab}$ be the conformal Killing operator, with $\mathbf{v}^a$ a smooth vector field on $\mathcal{M}$. The fields

$$\hat{h}_{ab} = \phi^4 h_{ab}, \quad \hat{\mathbf{j}}^{\,a} = \phi^{-10}\mathbf{j}^{\,a}, \quad \hat{\rho} = \phi^{-8}\rho,$$

$$\hat{k}^{ab} = \phi^{-10}\left[(\mathcal{L}\mathbf{w})^{ab} + \sigma^{ab}\right] + \frac{\tau}{3}\phi^{-4} h^{ab},$$

solve the Einstein constraints and energy cond. iff ϕ and $\mathbf{w}^a$ solve

$$-\Delta\phi + \frac{R}{8}\phi + \frac{\tau^2}{12}\phi^5 - \frac{[\sigma + (\mathcal{L}\mathbf{w})]^2}{8}\phi^{-7} - \frac{\kappa\rho}{4}\phi^{-3} = 0,$$

$$-\nabla_a(\mathcal{L}w)^{ab} + \frac{2}{3}(\nabla^b\tau)\phi^6 + \kappa\,\mathbf{j}^{\,b} = 0.$$

The conformal method to solve the constraint equations.

Remarks:

- The Conformal method: Lichnerowicz 1944, York 1972, [1].
- The undetermined constraint equations for the physical (hatted) fields $\hat{h}_{ab}$, $\hat{k}^{ab}$ are transformed into an elliptic system for ϕ and $\mathbf{w}^a$.
- $\hat{h}_{ab} = \phi^4 h_{ab}, \qquad \hat{k}^{ab} = \phi^{-10}\left[(\mathcal{L}\mathbf{w})^{ab} + \sigma^{ab}\right] + \frac{\tau}{3}\phi^{-4} h^{ab}$.
  These definitions imply: ϕ is the metric conformal factor, and $\mathbf{w}^a$ is the longitudinal part of of the extrinsic curvature.
- The field τ is the mean extrinsic curvature of the initial data. τ couples the two elliptic equations in the system.
- τ constant case is called constant mean curvature (CMC):

  $$-\Delta\phi + a_R\phi + a_\tau\phi^5 - a_w\phi^{-7} - a_\rho\phi^{-3} = 0, \; -\nabla_a(\mathcal{L}w)^{ab} + \mathbf{b}_j^b = 0.$$

  First solve the linear equation, then solve the equation for ϕ.

The problem of initial data in general relativity.

- A constrained Cauchy problem: Maxwell's equations.
- The problem of initial data for Maxwell's equations.
- The problem of initial data for Einstein's equations.
- The conformal method to solve the constraint equations.
- Results on closed manifolds:
  - **The CMC case.** (Understood by 1995.)
  - The near-CMC case.
  - The far-CMC case.

The CMC case: Understood by 1995.

Theorem (Isenberg, 1995, [3].)

If $\mathcal{M}$ is a smooth closed manifold, $h_{ab} \in C^3(\mathcal{M})$, $\sigma^{ab} \in W^{2,p}(\mathcal{M})$, $p > 3$, τ constant, $\rho = 0$, $\mathbf{j}^a = 0$, then the Lichnerowicz equation

$$-\Delta\phi + a_R\phi + a_\tau\phi^5 - a_\sigma\phi^{-7} = 0,$$

admits or does not admit a positive definite solution $\phi \in C^{2,\alpha}(\mathcal{M})$, with $\alpha \in (0,1)$, as indicated in the following table:

Yamabe class of h_{ab}	$\sigma^2 = 0$, $\tau = 0$	$\sigma^2 = 0$, $\tau \neq 0$	$\sigma^2 \neq 0$, $\tau = 0$	$\sigma^2 \neq 0$, $\tau \neq 0$
$\mathcal{Y}^+$	No	No	Yes	Yes
$\mathcal{Y}^0$	Yes	No	No	Yes
$\mathcal{Y}^-$	No	Yes	No	Yes

Idea of the proof: Maximum principles and barriers. □

The problem of initial data in general relativity.

- A constrained Cauchy problem: Maxwell's equations.
- The problem of initial data for Maxwell's equations.
- The problem of initial data for Einstein's equations.
- The conformal method to solve the constraint equations.
- Results on closed manifolds:
  - The CMC case. (Understood by 1995.)
  - **The near-CMC case.** (First result in 1996.)
    $|\nabla\tau| \ll 1$, Maximum principles, barriers, for ϕ;
    Linear elliptic theory for $\mathbf{w}^a$;
    And a contraction fixed-point argument
    for the coupled system, [4].
  - The far-CMC case.

The problem of initial data in general relativity.

- A constrained Cauchy problem: Maxwell's equations.
- The problem of initial data for Maxwell's equations.
- The problem of initial data for Einstein's equations.
- The conformal method to solve the constraint equations.
- Results on closed manifolds:
  - The CMC case. (Understood by 1995.)
  - The near-CMC case. (First result in 1996.)
  - **The far-CMC case.** (First results in 2008.)

The far-CMC case.

Recall the problem we want to solve:
Find ϕ and $\mathbf{w}^a$ solutions of the Lichnerowicz-York equations

$$-\Delta\phi + a_R\phi + a_\tau\phi^5 - a_w\phi^{-7} - a_\rho\phi^{-3} = 0,$$

$$-\nabla_a(\mathcal{L}w)^{ab} + \frac{2}{3}(\nabla^b\tau)\,\phi^6 + \mathbf{b}_j^b = 0.$$

when all the other fields are properly given on a closed manifold.

Definition
The smooth functions ϕ_- and ϕ_+ are *barriers* (sub- and super-solutions, respectively) iff holds

$$-\Delta\phi_- + a_R\phi_- + a_\tau\phi_-^5 - a_w\phi_-^{-7} - a_\rho\phi_-^{-3} \leqslant 0, \qquad (5)$$

$$-\Delta\phi_+ + a_R\phi_+ + a_\tau\phi_+^5 - a_w\phi_+^{-7} - a_\rho\phi_+^{-3} \geqslant 0. \qquad (6)$$

The barriers are *compatible* iff $0 < \phi_- \leqslant \phi_+$; they are *global* iff Eqs.(5)-(6) hold for all $\mathbf{w}^a$ solving York's equation $\forall\,\phi \in [\phi_-, \phi_+]$.

The far-CMC case.

Theorem (Host, Nagy, Tsogtgerel, 2008, [2].)

Let $(\mathcal{M}, h_{ab})$ be a smooth, closed, Riemannian manifold with $h_{ab} \in \mathcal{Y}^+$ and no conformal Killing vectors, $\tau \in W^{1,p}(\mathcal{M})$, $p > 3$, and σ^2, $\mathbf{j}^{\,a}$, $\rho \in L^p(\mathcal{M})$. If ϕ_-, ϕ_+ are compatible, global barriers to the Lichnerowicz equation, then there exist

$$\phi \in [\phi_-, \phi_+] \cap W^{2,p}(\mathcal{M}), \qquad \mathbf{w}^a \in W^{2,p}(\mathcal{M}),$$

solutions of the Lichnerowicz-York constraint equations.

The proof is based on:

- A version of Schauder Fixed-Point Theorem.
- Compact embedding $W^{2,p}(\mathcal{M}) \subset L^\infty(\mathcal{M})$.
- The order structure of $L^\infty(\mathcal{M})$.

The previous items hold without the condition $|\nabla\tau| \ll 1$. □

Remark: We only need compatible, global barriers $\phi_\pm$.

The far-CMC case.

Remark: Global super-solutions are harder to find than global sub-solutions.

Theorem (Global super-solution, [2].)

Let $(\mathcal{M}, h_{ab})$ be a smooth, closed, Riemannian manifold with $h_{ab} \in \mathcal{Y}^+$ and no conformal Killing vectors, and let the smooth function u be solution of the Yamabe problem

$$-\Delta u + \frac{R}{8}\, u - u^5 = 0.$$

Denote $k = \sup u / \inf u$, and let $K > 0$ a constant related to bounds in the norm of the conformal Killing op. $\mathcal{L} : W^{2,p} \to W^{1,p}$. If τ, σ^2, $\mathbf{j}^{\,a}$, ρ are all small enough, then

$$\phi_+ = \epsilon\, u, \qquad \epsilon = \Big(\frac{1}{k^{12} K}\Big)^{1/4}$$

is a global super-solution of the Lichnerowicz equation.

The problem of initial data in general relativity.

- A constrained Cauchy problem: Maxwell's equations.
- The problem of initial data for Maxwell's equations.
- The problem of initial data for Einstein's equations.
- The conformal method to solve the constraint equations.
- Results on closed manifolds:
  - The CMC case.
  - The near-CMC case.
  - The far-CMC case.

G. Cook.
Initial data for numerical relativity.
Living Rev. Relativity, 3(5), 2000.
http://www.livingreviews.org/lrr-2000-5.

M. Holst, G. Nagy, and G. Tsogtgerel.
Far-from-constant mean curvature solutions of Einstein's constraint equations with positive Yamabe metrics.
Phys. Rev. Lett., 100:161101, 2008.

J. Isenberg.
Constant mean curvature solution of the Einstein constraint equations on closed manifold.
Class. Quantum Grav., 12:2249–2274, 1995.

J. Isenberg and V. Moncrief.
A set of nonconstant mean curvature solution of the Einstein constraint equations on closed manifolds.
Class. Quantum Grav., 13:1819–1847, 1996.