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Relativistic Lagrange formulation

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It is well-known that the equations for a simple fluid can be cast into what is called their Lagrange formulation. We introduce a notion of a generalized Lagrange formulation, which is applicable to a wide variety of systems of partial differential equations. These include numerous systems of physical interest, in particular, those for various material media in general relativity. There is proved a key theorem, to the effect that, if the original (Euler) system admits an initial-value formulation, then so does its generalized Lagrange formulation. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1364502]

I. INTRODUCTION

Consider a simple perfect fluid in general relativity. That is, fix a space-time—a 4-dimensional manifold M with metric g_{ab} of Lorentz signature $(-,+,+,+)$. The fluid is described thereon by two fields, a unit timelike vector field u^a (which is interpreted as the velocity field of the fluid), and a scalar field ρ (which is interpreted as its mass density). These fields must satisfy the fluid equations,

$$(\rho+p)u^m\nabla_m u^a = -(g^{am}+u^a u^m)\nabla_m p, \tag{1}$$

$$\nabla_m(\rho u^m) = -p\nabla_m u^m. \tag{2}$$

Here p is specified as some fixed function of ρ, the function of state.

This treatment is usually called the Euler formulation of a fluid. Its characteristic feature is that the fluid is described by means of fields on space–time. That is, the ''independent variable'' in this formulation—the thing the fields are functions of—is the event of space-time. There is an alternative treatment of a fluid, called the Lagrange formulation, in which we ''move with the fluid, rather than remain fixed in space–time.'' In other words, the independent variable in this formulation is the fluid-element, and so the fluid is described by fields that are functions on the manifold of fluid-elements.[1]

Each of these two formulations has its advantages. The Euler formulation is less tightly tied down to the fluid itself, and so is usually more convenient when other systems—which would naturally be described with reference to space–time—are involved. In particular, the Euler formulation is normally used for a fluid in interaction with other fields, as, for example, in the Einstein-fluid system. The Lagrange formulation, by contrast, tends to be more convenient when one wishes to identify and follow individual fluid elements. For example, the Lagrange formulation might be used to describe a fluid object with a boundary. The boundary, in this formulation,

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would be fixed once and for all at the beginning (by designating those fluid-elements that constitute the boundary) as part of the kinematical structure. In the Euler formulation of such an object, by contrast, the boundary would be ''dynamical.''

How are the Euler and Lagrange formulations related to each other? Certainly, the two are physically equivalent, i.e., they represent mere mathematical reformulations of the same physics. That is, all physical predictions will be the same, no matter which formulation is used; and, at least in principle, either formulation could be used to solve any given problem. Indeed, one might be tempted to go further than this, to view them as related by a mere coordinate transformation on the manifold of independent variables. But such a viewpoint would be misleading, for the ''coordinate transformation'' between the two sets of variables involves the dynamics of the system. Thus, for example, from the standpoint of the Euler formulation the Lagrange formulation represents a curious mixing of kinematics with dynamics.

These mathematical differences in fact go even deeper. It is well-known that the equations for a perfect fluid in the Euler formulation, Eqs. (1)–(2), have a well-posed initial-value formulation.[3] But the corresponding equations in the Lagrange formulation—at least, those obtained directly, by simply ''transforming'' the Euler equations—do not.[5] However, it has been shown by Friedrich, in Ref. 17, that, at least for a certain fluid system in general relativity, there *can* be introduced a Lagrange formulation having also an initial-value formulation. It is necessary, in Friedrich's treatment, to introduce a substantial number of additional fields (including a frame-field) together with additional equations on those fields. What is not so transparent, however, is the mechanism behind this treatment. Precisely what features of these fluid systems are needed for such a deterministic Lagrange formulation?

Our purpose in this paper is to introduce and explore a certain, broad, geometrical setting for the Lagrange formulation of systems of partial differential equations.

In Sec. II, we introduce that setting. Our framework is systems of partial differential equations that are first-order and quasi-linear (i.e., involving only first derivatives of the fields, and those only linearly)—a framework that includes virtually every partial differential equation in physics. Given any such system—provided only that it has among its fields a distinguished vector field—we write out a new system, its ''Lagrange formulation.'' The key idea of this scheme is what one might expect: Include, among the dynamical variables of the new system, what were the independent variables of the original system. It turns out that, in order to execute this scheme, it is normally necessary to introduce additional dynamical variables and equations. We give a general scheme for choosing these variables. The key result of this section is the following: Given any system of partial differential equations having a distinguished vector field as above, and also having an initial-value formulation, then a certain version of its Lagrange formulation also has an initial-value formulation.

In Sec. III, we give some examples of this scheme. We apply the present scheme not only to ordinary fluids, but also to various other types of material systems, including dissipative fluids and elastic solids. This scheme is also applicable when such material systems are undergoing interaction, e.g., when they are coupled to an electromagnetic or gravitational field. Finally, we show in Sec. III how Friedrich's original system fits within the present framework.

A number of related mathematical issues are discussed in the appendices. In Appendix A, we describe a general procedure for modifying any system of partial differential equations by ''taking derivatives'' of the fields of that system. This procedure, it turns out, is crucial for casting systems into a form in which our Lagrange formulation can be applied. In Appendix B, we review a few facts about the initial-value formulation of systems of partial differential equations. (For a more detailed treatment, see, for example, Ref. 2.)

II. LAGRANGE FORMULATION

Fix a first-order, quasilinear system of partial differential equations. That is, let there be given a fiber bundle, consisting of some base manifold M, some bundle manifold $\mathcal{B}$, and some smooth projection mapping $\mathcal{B} \xrightarrow{\pi} M$. Typically, M will be the 4-dimensional manifold of space–time events (but it could be any smooth manifold). By the *fiber* over a point x of M, we mean the set of all points y of $\mathcal{B}$ such that $\pi(y)=x$. Think of the fiber over $x\in M$ as ''the set of possible field-values at x.'' Then $\mathcal{B}$ is interpreted as the set of ''all possible choices of field-values at all points of M,'' and π as the mapping that assigns, to each such choice, the underlying point of M. Thus, point y of $\mathcal{B}$ could be written as $y=(x,\phi)$, with $x\in M$ and ϕ in the fiber over x. The action of the projection mapping would then be given by $\pi(x,\phi)=x$. Typically, the fiber over a point $x\in M$ will be some collection of tensors, with given index structure (possibly subject to various algebraic conditions), at x, whence $\mathcal{B}$ will be a manifold of all such tensor-collections at all points of M. In this case, $\mathcal{B}$ is called a *tensor bundle*. However, $\mathcal{B}$ could in general be any smooth manifold, subject only to the local-product condition in the definition of a fiber bundle.[6]

By a *cross-section* of such a bundle we mean a smooth mapping $M\xrightarrow{\phi}\mathcal{B}$ such that $\pi\circ\phi$ is the identity map on M. In other words, a cross-section assigns, to each point x of M, a point of the fiber over x; i.e., it assigns a ''field-value'' at each point of M. In the case of a tensor bundle, a cross-section is simply a certain collection of smooth tensor fields on M. Our partial differential equation will be an equation on this map, linear in its first derivative. In order to write out this equation, we introduce two smooth fields, $k^{Aa}{}_{\alpha}$ and j^A, on $\mathcal{B}$. Since these are fields on $\mathcal{B}$, they depend on point $y=(x,\phi)$ of $\mathcal{B}$, i.e., they depend on a choice of ''point x of the base manifold, as well as field-value ϕ at that point.'' The index ''α'' on $k^{Aa}{}_{\alpha}$ is a tensor index in $\mathcal{B}$ at the point, $y\in\mathcal{B}$, at which this field is evaluated; the index ''a'' is a tensor index in M at the corresponding point, $\pi(y)$, of the base manifold. The index ''A,'' on both $k^{Aa}{}_{\alpha}$ and j^A, lies in some new vector space (which will turn out, shortly, to be the vector space of equations). Finally, our partial differential equation, on a cross-section ϕ, is

$$k^{Aa}{}_{\alpha}(\nabla\phi)_a{}^{\alpha}=j^A. \tag{3}$$

This equation is to be imposed at each point $x\in M$, with the fields k and j evaluated at $\phi(x)\in\mathcal{B}$, i.e., on the cross-section. Here, $(\nabla\phi)_a{}^{\alpha}$ denotes the derivative of the map ϕ [i.e., a map from tangent vectors in M at x to tangent vectors in $\mathcal{B}$ at $\phi(x)$]. The index ''A'' in Eq. (3) is free, i.e., Eq. (3) represents a number of scalar equations equal to the dimension of the vector space in which ''A'' lies.

Here is an example. Fix a 4-dimensional manifold M, together with a Lorentz-signature metric g_{ab} on this M. Let $\mathcal{B}$ be the 8-manifold consisting of triples, (x,u^a,ρ), where x is a point of M, u^a is a unit timelike vector at x, and ρ is a number. Let $\pi(x,u^a,\rho)=x$. This is a fiber bundle; in fact, a tensor bundle. The fiber over a point $x\in M$ consists of (u^a,ρ), a vector at x together with a number. A cross-section of this bundle is represented by smooth fields, u^a and ρ, on M. Let the equations, on such a cross-section, be (1)–(2), where $p(\rho)$ is some given, fixed function of one variable, and ∇_a is the derivative operator defined by the space–time metric g_{ab}. This is a first-order, quasilinear system of partial differential equations, i.e., the equations are linear in the first derivatives of the fields. The vector space of equations, in this example, has dimension four. This system, of course, describes a simple perfect fluid in general relativity.

We shall now introduce a technique that transforms a given first-order, quasilinear system of partial differential equations—provided that system lies within a certain class—into a new first-order, quasilinear system of partial differential equations. This new system will be called the *Lagrange formulation* of the original. While the new system will differ in many respects from the original one—e.g., it will have a different base manifold, a different bundle manifold, and a different number of equations—the two will be intimately related to each other. In particular, it will turn out that there is a natural correspondence between the solutions of the original system and those of its Lagrange formulation.

In order to apply this technique to a given system of equations, it is necessary that that system satisfy the following condition: Among the various fields of the system there must be distinguished one consisting of a nowhere-vanishing vector field on the base manifold M. This condition means, then, that the fields of our system take the form (u^a,φ), where u^a represents the nowhere-vanishing vector field on M, and φ represents ''the rest of the fields.'' Thus, given a system that has, among its various fields, no vector field at all, then we shall be unable to write out any Lagrange formulation for it; and if it has several vector fields, then we must, at this stage, distinguish a particular one. We shall denote by $B \xrightarrow{\pi} M$ the bundle in which the rest of the fields, the φ, lie, and use Greek indices for tensors in the manifold B. Note that these are *different* from the Greek indices, e.g., in Eq. (3), for tensors in the manifold $\mathcal{B}$. The equation for our system may now be written as

$$k'^{Aa}{}_b \nabla_a u^b + k''^{Aa}{}_\alpha (\nabla \varphi)_a{}^\alpha = j^A, \tag{4}$$

where $k'^{Aa}{}_b$, $k''^{Aa}{}_\alpha$, and j^A are all functions of u^a, φ, and point of M. In Eq. (4), the ∇_a in the first term can be any derivative operator on M; and the form of j^A depends, of course, on what operator has been chosen. We could, for example, simply fix, once and for all, some derivative operator ∇_a, and use it to write Eq. (4). Should it happen that the manifold M comes equipped with a kinematical metric (i.e., one not included among the physical fields φ), then it is often convenient to use its derivative operator in Eq. (4). This possibility is available, e.g., for systems representing fluids in special relativity, or in general relativity with a fixed background metric. In fact, we could even choose the derivative operator ∇_a in Eq. (4) to depend on the fields (u^a, φ) themselves, provided only that its dependence on these fields is algebraic, rather than through their derivatives. We now obtain the Lagrange formulation of this system.

For the base manifold of the Lagrange formulation, we choose any manifold $\hat{M}$ having the same dimension as M. Tensors over this $\hat{M}$ will be denoted by lower-case Latin indices with hats. We also fix, once and for all on this manifold $\hat{M}$, a nowhere-vanishing vector field, $\hat{u}^{\hat{a}}$. This $\hat{u}^{\hat{a}}$ is a purely kinematical object, i.e., it is fixed right at the beginning, and will not be subject to any dynamical equations.

We next specify the bundle manifold, $\hat{\mathcal{B}}$, of the Lagrange formulation. Fix a point, $\hat{x}$, of the base manifold $\hat{M}$. Let the fiber over this point consist of a triple, $(x, \varphi, \kappa_{\hat{a}}{}^b)$, where (i) x is a point of M, the base manifold of the original system, (ii) φ is a point of the fiber over x in B, the bundle manifold for the original system, and (iii) $\kappa_{\hat{a}}{}^b$ is an invertible tensor, where the index ''$\hat{a}$'' refers to tensors in $\hat{M}$ at the point $\hat{x} \in \hat{M}$ and the index ''b'' refers to tensors in M at the point $x \in M$. A more detailed discussion of these three objects follows.

(i) The points (x) of the base manifold M of the original system become, in its Lagrange formulation, *field-values*. In the case of a simple perfect fluid, for example, each point of the original base manifold M represents an event of space–time; while each point of the new base manifold $\hat{M}$ represents ''a particular fluid-element at a particular moment of its life.'' Thus, in the Lagrange formulation of such a fluid, x will be a field over $\hat{x}$, a field that specifies ''which event in space–time that particular fluid-element occupies at that particular moment.''

(ii) The field-values, the φ, of the original system become field-values also in its Lagrange formulation. But there is one important change: What were fields over M in the original system become, in its Lagrange formulation, fields over $\hat{M}$. Thus, were the fields collected in φ all tensor fields on M, then the corresponding fields in the Lagrange formulation would depend on point $\hat{x}$ of $\hat{M}$, but would continue to be tensors in the tangent space at the point x of M.[7] In the case of a simple perfect fluid, this step amounts, physically, to ''attaching the density ρ to the fluid element, rather than to the point of space–time.''

(iii) There is introduced a new object, $\kappa_{\hat{a}}{}^b$, an invertible two-point tensor, with one index at $\hat{x} \in \hat{M}$, the other at $x \in M$. Nothing analogous was present in the original system. Denote the inverse of $\kappa_{\hat{a}}{}^b$ by $\overline{\kappa}_b{}^{\hat{a}}$, so we have $\kappa_{\hat{a}}{}^b \overline{\kappa}_b{}^{\hat{c}} = \delta_{\hat{a}}{}^{\hat{c}}$ and $\overline{\kappa}_b{}^{\hat{a}} \kappa_{\hat{a}}{}^c = \delta_b{}^c$. The role of this tensor $\kappa_{\hat{a}}{}^b$ is, as we shall see, to preserve the first-order character of the final system of equations. Note that the dynamical field u^a in the original system has disappeared entirely: There is no analog of it as a dynamical field in the Lagrange formulation.

Next note that the pair (x,φ), where x is a point of M and φ is a point of the fiber in B over x, is precisely the same thing as a point of the bundle manifold B. Call that point (for later convenience) $\hat\varphi$, so we have $\hat\varphi=(x,\varphi)\in B$. Then we may recover the point x of the original base-manifold from the point $\hat\varphi\in B$ using the projection π: We have $x=\pi(\hat\varphi)$. Thus, our construction of the bundle manifold $\hat{\mathcal{B}}$ for the Lagrange formulation could have been stated as follows: The fiber over point $\hat x\in\hat M$ consists of a pair, $(\hat\varphi,\kappa_{\hat a}{}^b)$, where $\hat\varphi$ is a point of the manifold B, and $\kappa_{\hat a}{}^b$ is an invertible tensor with one index at $\hat x\in\hat M$, the other at $\pi(\hat\varphi)\in M$.

We have now completed the specification of the fiber bundle in which the Lagrange formulation of our system will be written. The base manifold, $\hat M$, is some new manifold, of the same dimension as M, while the bundle manifold $\hat{\mathcal{B}}$ is such that the fiber over $\hat x\in\hat M$ consists of a pair, $(\hat\varphi,\kappa_{\hat a}{}^b)$, where $\hat\varphi\in B$, and $\kappa_{\hat a}{}^b$ is a certain 2-point tensor. A cross-section of this bundle, then, is a smooth map (a map we also denote by $\hat\varphi$) that assigns, to each point $\hat x\in\hat M$, a point $\hat\varphi$ of B together with a suitable tensor $\kappa_{\hat a}{}^b$. On such a cross-section, we now impose the following equations:

$$(\nabla(\pi\circ\hat\varphi))_{\hat a}{}^b=\kappa_{\hat a}{}^b, \tag{5}$$

$$\nabla_{[\hat c}(\kappa_{\hat a]}{}^b)=f_{\hat c\hat a}{}^b, \tag{6}$$

$$k'^{Aa}{}_b\overline{\kappa}_a{}^{\hat d}\nabla_{\hat d}(\kappa_{\hat c}{}^b\hat u^{\hat c})+k''^{Aa}{}_\alpha\overline{\kappa}_a{}^{\hat c}(\nabla\hat\varphi)_{\hat c}{}^\alpha=j^A. \tag{7}$$

These are the equations of the Lagrange formulation. In Eq. (5), the combination $\pi\circ\hat\varphi$ is a map from $\hat M$ to M, for $\hat\varphi$ goes from $\hat M$ to B, and π from B down to M. Equation (5) asserts that the derivative of this map is precisely the tensor $\kappa_{\hat a}{}^b$. Thus, this equation provides the geometrical meaning of the field $\kappa_{\hat a}{}^b$. Note that invertibility of $\kappa_{\hat a}{}^b$ in Eq. (5) implies that the map $\pi\circ\hat\varphi$ from $\hat M$ to M is a local diffeomorphism between these two manifolds. It was to achieve this feature that we originally choose $\hat M$ to have the same dimension as M. Equation (6) is merely the curl[8] of Eq. (5). Any derivative[9] may be used on the left in Eq. (6), but the exact form of the function $f_{\hat c\hat a}{}^b$ [of $(\hat\varphi,\kappa_{\hat a}{}^b)$] that appears on the right will depend on which derivative was chosen. This situation is analogous to that of Eq. (4). Equation (7) is the translation of the equation of the original system (4) to our new system. Here, everywhere in the fields $f_{\hat c\hat a}{}^b$, $k'^{Aa}{}_b$, $k''^{Aa}{}_\alpha$, and j^A there is to be substituted the combination ''$\kappa_{\hat a}{}^b\hat u^{\hat a}$'' for ''$u^b$;'' and ''$\hat\varphi$'' for ''$\varphi$.'' In Eq. (7), this ''replacement'' takes place even inside the derivative. Note that the field u^b of the original system has now disappeared entirely, having been replaced by the image of the kinematical field $\hat u^{\hat b}$ under the mapping $\pi\circ\hat\varphi$.

Thus, beginning with any first-order, quasilinear system of partial differential equations of the form (4), we obtain a new system of equations, its Lagrange formulation, of the form (5)–(7). The Lagrange formulation has a completely new base space, but fields and equations that echo those of the original system.

We now claim the following: Every solution of the Lagrange formulation gives rise, at least locally, to a solution of the original system. Indeed, let $(\hat\varphi,\kappa_{\hat a}{}^b)$ be fields satisfying (5)–(7). Then, as we have seen, $\pi\circ\hat\varphi$ is a local diffeomorphism between $\hat M$ and M. We now introduce the following two fields on M: $u^b=(\nabla(\pi\circ\hat\varphi))_{\hat a}{}^b\hat u^{\hat a}$, and $\varphi=\hat\varphi\circ(\pi\circ\hat\varphi)^{-1}$. That is, we let u^b and φ be the images of $\hat u^{\hat b}$ and $\hat\varphi$, respectively, under the diffeomorphism $\pi\circ\hat\varphi$. Then these fields, (u^b,φ), on M satisfy the system (4), as is immediate from Eqs. (5), (7). We next claim that the converse also holds: Every solution of the original system gives rise, at least locally, to a solution of its Lagrange formulation. Indeed, let (u^b,φ) be fields satisfying (4). Choose any manifold $\hat M$ with the same dimension as that of M, and any nowhere-vanishing vector field $\hat u^{\hat a}$ thereon. Now let $\hat\varphi$ be a diffeomorphism between $\hat{M}$ and the cross-section, $\varphi[M]$, such that $(\pi\circ\hat{\varphi})$ sends $\hat{u}$ to u; and then define $\kappa_{\hat{a}}{}^{b}$ by Eq. (5). Then these fields $(\hat{\varphi},\kappa_{\hat{a}}{}^{b})$ on $\hat{M}$ will satisfy Eqs. (5)–(7) [the first two by construction, the last by Eq. (4)].

Thus, the original system and its Lagrange formulation are identical as to solutions. But the two systems are quite different as to form. Their base manifolds, $\hat{M}$ and M, although of the same dimension, differ in their geometry. The manifold $\hat{M}$ must be endowed with a fixed, kinematical ''velocity field,'' $\hat{u}^{\hat{a}}$, while M has no such kinematical field. On the other hand, various kinematical fields that might have been specified over M (such as a Lorentz metric) yield no analogous kinematical fields[10] on $\hat{M}$. Furthermore, the fields of the two systems differ in several respects. Beginning with the fields of the original system, we must delete the dynamical field u^a, while adding ''point of M'' as well as the invertible tensor $\kappa_{\hat{a}}{}^{b}$, to obtain the fields of the Lagrange formulation. Finally, the equations for the two systems differ in that, for the Lagrange formulation, there must be introduced one new equation (5) on the derivative of the ''point of M,'' as well as is the curl (6) of this new equation.

What we have described above is precisely what is usually done in writing down the Lagrange formulation for a fluid. For example, consider again the simple perfect fluid, with fields (u^a,ρ) on M and Eqs. (1)–(2). Its Lagrange formulation consists of fields[11,12] $(x,\kappa_{\hat{b}}{}^{a},\hat{\rho})$ on $\hat{M}$, with equations consisting of (5), (6), and

$$(\hat{\rho}+p(\hat{\rho}))\hat{u}^{\hat{c}}\nabla_{\hat{c}}(\kappa_{\hat{m}}{}^{a}\hat{u}^{\hat{m}})+(g^{am}+\hat{u}^{\hat{c}}\kappa_{\hat{c}}{}^{a}\hat{u}^{\hat{n}}\kappa_{\hat{n}}{}^{m})\overline{\kappa_m}^{\hat{b}}\nabla_{\hat{b}}p(\hat{\rho})=0, \tag{8}$$

$$\hat{u}^{\hat{b}}\nabla_{\hat{b}}\hat{\rho}+(\hat{\rho}+p(\hat{\rho}))\overline{\kappa_a}^{\hat{b}}\nabla_{\hat{b}}(\kappa_{\hat{m}}{}^{a}\hat{u}^{\hat{m}})=0. \tag{9}$$

We now return to the general case. It turns out that the procedure given above—starting with a system and ending with its Lagrange formulation—suffers from a serious difficulty. In general, the equations of the Lagrange formulation, (5)–(7), will fail to have an initial-value formulation, even if the original system, (4), did have such a formulation. For example, the system (5)–(6), (8)–(9) has no initial-value formulation, although the system (1)–(2) of course does. But it turns out that this difficulty does not arise—i.e., the Lagrange formulation does inherit an initial-value formulation from the original system—provided the original system satisfies the following condition: There can be derived from Eq. (4) an expression for the derivative of the vector field u^a, without contractions, back in terms of the various fields of the system. In other words, it must be possible to cast Eq. (4) into the form

$$\nabla_a u^b = w_a{}^b, \tag{10}$$

$$k''^{Aa}{}_{\alpha}(\nabla\varphi)_a{}^{\alpha}=j'^{A}, \tag{11}$$

where $w_a{}^b$, $k''^{Aa}{}_{\alpha}$, and j'^{A} are functions of (x,u^a,φ), i.e., are functions of the point of B and the vector u^a. In Eq. (10), ∇_a can, again, be any derivative operator on the manifold M; and the form of $w_a{}^b$ depends, of course, on what operator has been chosen. Note that, once we have derived from Eq. (4) an equation of the form (10), then it is easy to cast the equations that remain into the form (11): Simply use Eq. (10) to remove all u-derivatives from Eq. (4). Indeed, we have $j'^{A} = j^A - k'^{Aa}{}_b w_a{}^b$.

The equations for systems of physical interest typically do *not* take the form of Eqs. (10)–(11), i.e., they do not express the derivative of u^a in terms of the other fields. For example, Eqs. (1)–(2) do not have this form. But it turns out that there is a simple, general procedure by which *any* first-order, quasilinear system of partial differential equations having a preferred vector field u^a can be recast so as to take the form (10)–(11). This procedure, called taking the *derivative system*, is spelled out in Appendix A. It consists of modifying the original system by introducing additional fields, which represent the derivatives of the original fields, as well as additional equations on those fields. The result of taking the derivative system is to produce a new system of partial differential equations, having, in an appropriate sense, identical solutions to the original. Applied to a system in which a preferred vector field u^a has been distinguished, it produces a system in which $\nabla_a u^b$ is expressed back in terms of the fields of the system. Furthermore, applied to any system having an initial-value formulation, the derivative system also has an initial-value formulation.

As an example of this procedure, we return to the system, (1)–(2), for a simple perfect fluid in general relativity. For the distinguished nowhere-vanishing vector field in this case, we choose, of course, the velocity field u^a of the fluid. The result of taking the derivative system of this system is the following. The fields consist of $(u^a, \rho, w_a{}^b, v_a)$, where u^a is a unit timelike vector field, ρ a positive scalar field, $w_a{}^b$ a tensor field satisfying $g_{ab}u^a w_c{}^b = 0$, and v_a a vector field, all subject to the algebraic conditions

$$(\rho+p)u^m w_m{}^a + (g^{am}+u^a u^m)(\partial p/\partial\rho)v_m = 0, \tag{12}$$

$$u^m v_m + (\rho+p)w_m{}^m = 0. \tag{13}$$

On these fields is imposed the following system of first-order, quasilinear partial differential equations:

$$\nabla_a u^b = w_a{}^b, \tag{14}$$

$$\nabla_{[a} w_{b]}{}^c = R_{abm}{}^c u^m, \tag{15}$$

$$\nabla_a \rho = v_a, \tag{16}$$

$$\nabla_{[a} v_{b]} = 0. \tag{17}$$

Note what has happened here. We have introduced two new fields, $w_a{}^b$ and v_a. The ''interpretation'' of $w_a{}^b$ [via (14)] is as the derivative of u^b; and of v_a [via (16)] as the derivative of ρ. The original fluid equations, (1)–(2), have been converted into algebraic conditions, (12)–(13), on these new fields. That is, the original fluid equations serve merely to define the bundle of fields for this new system. Finally, the new system contains two other equations, Eqs. (15) and (17), that are merely the curls of Eqs. (14) and (16), respectively.

In short, our ''procedure'' has done nothing of substance. But note that, starting with a system (1)–(2), which fails to express $\nabla_a u^b$ in terms of the fields of the system, our procedure produces a new system satisfying, via (14), this condition. Furthermore—and this is perhaps the striking feature—the system (14)–(17) inherits from the original fluid system, (1)–(2), its initial-value formulation.

The key result of this section is the following: *Consider any system (4) of partial differential equations in which there has been selected a preferred vector field u^a. Let (i) that system have an initial-value formulation, and (ii) the equations of that system express the derivative of u^a in terms of the fields of the system [as in (10)–(11)]. Then the Lagrange formulation of that system also admits an initial-value formulation*.

First note that the Lagrange formulation of the system (10)–(11) consists of Eqs. (5)–(6), together with

$$\overline{\kappa_a}^{\hat{c}} \nabla_{\hat{c}}(\kappa_{\hat{m}}{}^b \hat{u}^{\hat{m}}) = w_a{}^b, \tag{18}$$

$$k''^{Aa}{}_\alpha \overline{\kappa_a}^{\hat{b}} (\nabla\hat{\varphi})_{\hat{b}}{}^\alpha = j'^A. \tag{19}$$

As discussed in Appendix B, in order that a general first-order, quasilinear system of partial differential equations have an initial-value formulation it is necessary that it satisfy three conditions: (i) the system admits a hyperbolization; (ii) all the constraints of the system are integrable, and (iii) the system has the correct number of equations relative to the number of its unknowns. What these conditions mean is also explained in Appendix B. We check these three conditions in turn.

Let the original system, Eqs. (10)–(11), admit a hyperbolization. Then the construction that, applied to Eqs. (10)–(11) to obtain a bilinear expression in $\delta\varphi^{\alpha}$ yields, when applied to Eqs. (18)–(19), a corresponding bilinear expression in $\delta\hat{\varphi}^{\hat{\alpha}}$. Next, contract Eq. (6) with $\hat{u}^{\hat{c}}$ and use Eq. (18) to obtain an equation expressing $\hat{u}^{\hat{m}}\nabla_{\hat{m}}\,\kappa_{\hat{a}}{}^{b}$ algebraically in terms of the fields. From this there follows immediately an appropriate bilinear expression in $\delta\kappa_{\hat{a}}{}^{b}$. Finally, a bilinear expression in δx arises from Eq. (5). These three bilinear expressions, taken together, represent a hyperbolization for the system (5)–(6), (18)–(19).

Every constraint of the original system (10)–(11) gives rise to a constraint of its Lagrange formulation; and, furthermore, if these constraints of the original system are integrable, then so are the corresponding constraints of the Lagrange formulation.[13] This assertion is immediate from the fact that Eqs. (18) and (19) mimic Eqs. (10) and (11), respectively. But, it turns out, there are two additional classes of constraints for the system of the Lagrange formulation. The first class arises from taking the curl of each side of Eq. (5). These constraints are certainly integrable, and, indeed, the corresponding integrability conditions are precisely Eq. (6). The second class of constraints arises from taking the curl of each side of Eq. (6). These constraints are also integrable, and indeed their integrability conditions are identities, simply from the way Eq. (6) was obtained. We conclude, thus, that a system of the form (10)–(11) having all its constraints integrable leads to a Lagrange formulation (5)–(6) (18)–(19), also having all its constraints integrable.

Finally, in order to check the third condition, we introduce the following integers. Denote by n the dimension of the base space M (the number of independent variables of the system), by u the dimension of the fibers in the bundle B (the number of unknowns represented by φ), by e the dimension of the vector space in which the index ''A'' of Eq. (11) lies, and by c the dimension of the space of vectors of the form $w_m c^m{}_A$, as $c^m{}_A$ runs over constraints for Eq. (11). Then, for the original system, we have the number of unknowns is given by $u_0=u+n$ (the term ''n'' arising from the field u^a); the number of equations is given by $e_0=n^2+e$ [these terms arising from Eqs. (10) and (11), respectively]; and the number of effective constraints is given by $c_0=n(n-1)+c$ [these terms arising from the constraints of Eqs. (10) and (11), respectively]. For the Lagrange formulation, on the other hand, we have: the number of unknowns is given by $u_L=u+n+n^2$ (the term ''n'' arising from the field ''point of M,'' the term ''n^2'' from the field $\kappa_{\hat{a}}{}^{b}$); the number of equations is given by $e_L=n^2+n^2(n-1)/2+n^2+e$ [these terms arising from Eqs. (5)–(6), (18)–(19), respectively]; and the dimension of the space of effective constraints is given by $c_L=n(n-1)+n(n-1)(n-2)/2+n(n-1)+c$ [these terms arising from the constraints of Eqs. (5)–(6), (18)–(19), respectively]. It is easy to check from these formulas that $e_0-c_0=u_0$ implies $e_L-c_L=u_L$. In other words, if the original system has the appropriate number of equations relative to its number of unknowns, then so does its Lagrange formulation.

Thus, we have shown a system of the form (10)–(11) having an initial-value formulation gives rise to a Lagrange formulation also with an initial-value formulation.

III. EXAMPLES

In this section, we introduce various examples of physical systems, the partial differential equations that describe them, and the Lagrange formulations of those partial differential equations.

One such example, the simple perfect fluid, has been discussed already in Sec. II. The fields, on space–time, M,g_{ab}, consist of a unit timelike vector field u^a (interpreted as the fluid velocity) and a positive scalar field ρ (interpreted as the mass density); and the equations are (1)–(2), where $p(\rho)$ is some fixed function (the function of state), which specifies the type of fluid under consideration. This is the Euler formulation. In order to achieve a Lagrange formulation for this system, the first step is to modify these equations so that the derivative of u^a, without contractions, is expressed in terms of the other fields. This was achieved by taking the derivative system: We introduced two new (tensor) fields, $w_a{}^b$ and v_a, subject to the algebraic conditions (12)–(13). We then imposed on the total set of fields, $(u^a,\rho,w_a{}^b,v_a)$, the partial differential equations (14)–(17). This new system (14)–(17) is, by virtue of Eq. (14), of the required form, and, in addition, it inherits from the original system, (1)–(2), its initial-value formulation. To this system, (14)–(17), we may therefore apply the methods of Sec. II to obtain its Lagrange formulation. There result fields $(x,\hat{\rho},\hat{w}_a{}^b,\hat{v}_a,\kappa_{\hat{a}}{}^b)$ on $\hat{M}$, subject to the equations (5)–(7). This new system, as demonstrated in Sec. II, again has an initial-value formulation.

There is a natural generalization of this simple perfect-fluid system to a much broader class of fluids. Fix some smooth manifold S, the points of which will, shortly, be interpreted as representing ''local, internal, states of the fluid.'' Also fix any space–time (M,g_{ab}). Let the fields, on this space–time, consist of a unit, timelike vector field, u^a (again interpreted as the velocity field of the fluid), together with a second field, φ, which is valued in S (and which is interpreted as giving the local state of the fluid at each point of space–time). Thus, φ is a mapping, $M\xrightarrow{\varphi}S$. As an example, the simple perfect-fluid system discussed above is the special case in which S is a 1-manifold (whose points are labeled by a coordinate ρ, whence φ reduces to the density field ρ). That is, our simple perfect fluid is one whose local state is completely characterized by the value of the density.

We next wish to write equations on these fields. To this end, fix two tangent vector fields, V^α and T^α, and one covector field, F_α, on the manifold S, where we have introduced Greek indices[14] to represent tensors in S. The physical interpretations of these fields will be given shortly. Let the equations for this system be

$$u^a\nabla_a u^b+(g^{ab}+u^a u^b)(\nabla\varphi)_a{}^\alpha F_\alpha=0, \tag{20}$$

$$u^a(\nabla\varphi)_a{}^\alpha+V^\alpha\,\nabla_a u^a+T^\alpha=0. \tag{21}$$

The first equation gives the fluid acceleration in terms of the derivative of the fluid state. We may interpret the field F_α, which acts by driving the fluid, as an ''effective force.'' The second equation gives the time rate of change of the fluid state in terms of that state and the divergence of u^a.[15] We may interpret the fields V^α and T^α, respectively, as giving the rate of change of fluid state under small volume-changes of a sample of that fluid, and under allowing a sample of that fluid to evolve in time. The simple perfect fluid, for example, has $F_\alpha=(\rho+p)^{-1}\nabla_\alpha p$, $V^\alpha=(\rho+p)\partial/\partial\rho$, and $T^\alpha=0$ [for these choices reproduce Eqs. (1)–(2)]. Another familiar example is the perfect fluid with 2-dimensional manifold S of internal states, where the additional degree of freedom is represented by a conserved particle-number n. In this case, F_α is given by the same expression as above, V^α by $(\rho+p)\partial/\partial\rho|_n+n\,\partial/\partial n|_\rho$, and again T^α by 0. A more exotic example is that of a fluid consisting of several species of particles, between which chemical reactions can take place as the fluid evolves. In this case, we would have $\dim(S)>2$ (the additional degrees of freedom describing the chemical composition of the fluid) and T^α nonzero (representing the rate and direction of the chemical reactions).

When does the system above satisfy the three properties, as discussed in Appendix B, for having an initial-value formulation? Two of these properties are immediate: Clearly, this system has no constraints, and the dimension of its space of equations is the same [namely, $\dim(S)+3$] as the dimension of its space of fields. As for the third condition, this system, it turns out, admits a hyperbolization if and only if[16] $V^\alpha F_\alpha>0$ everywhere on S. Note that, in the explicit examples given above, the combination $V^\alpha F_\alpha$ is precisely the square of the sound speed.

We now have a system of equations (20)–(21), having a preferred vector field, u^a, and, subject only to the inequality $V^\alpha F_\alpha>0$, having an initial-value formulation. So, we may apply to this system the results of Appendix A and Sec. II. The first step is to take the derivative system (Appendix A). The result of this step is to include, in addition to the fields u^a,φ above, two new fields, $w_a{}^b$ (with $u_b w_a{}^b=0$) and $\zeta_a{}^\alpha$, subject to the algebraic conditions $u^a w_a{}^b+(g^{ab}+u^a u^b)\zeta_a{}^\alpha F_\alpha=0$ and $u^a\zeta_a{}^\alpha+V^\alpha w_a{}^a+T^\alpha=0$. [These algebraic conditions reflect Eqs. (20)–(21).] The equations on these fields for the derivative system are given by

$$\nabla_a u^b = w_a{}^b, \tag{22}$$

$$\nabla_{[a} w_{b]}{}^c = R_{abd}{}^c u^d, \tag{23}$$

$$(\nabla \varphi)_a{}^\alpha = \zeta_a{}^\alpha, \tag{24}$$

$$\nabla_{[a} \zeta_{b]}{}^\alpha = 0. \tag{25}$$

This system indeed has a preferred vector field, u^a; has among its equations one [Eq. (22)] that expresses the derivative of this u^a algebraically in terms of the fields; and has an initial-value formulation [by virtue of that for Eqs. (20)–(21)]. So, we may, as described in Sec. II, take the Lagrange formulation of this system. There results a new system of partial differential equations, (5)–(7), again having an initial-value formulation.

Even the broad class of generalized fluids above does not include all possible types. For example, there exist fluids manifesting dissipative effects, such as heat-flow and viscosity. One description of such a fluid in relativity (Refs. 18–21) proceeds as follows. The fields consist of a unit timelike vector field u^a (interpreted as the fluid 4-velocity), two scalar fields, ρ and n (interpreted, respectively, as the fluid mass density and particle-number density), a vector field q_a satisfying $u^a q_a = 0$ (interpreted as the heat-flow vector), and a symmetric tensor field τ_{ab} satisfying $u^a \tau_{ab} = 0$ (interpreted as the stress tensor). Thus, the space of field-values at each point of M is 14-dimensional. The equations on these fields consist of (i) vanishing of the divergence of nu^a (conservation of particle number), (ii) vanishing of the divergence of $(\rho + p) u^a u^b + p g^{ab} + 2u^{(a} q^{b)} + \tau^{ab}$ (conservation of stress-energy), and (iii) a certain system of nine additional equations that, effectively, governs the dynamical evolution of q^a and τ^{ab}. It turns out that the resulting system, consisting of (i)–(iii), has an initial-value formulation: Specifically, it has a hyperbolization and no constraints. Furthermore—and this is perhaps surprising—this system of equations can be so chosen that it reduces, in an appropriate limit, to the familiar Navier–Stokes system for a dissipative fluid. [The Navier–Stokes dissipation coefficients—the thermal conductivity and viscosity—arise from within the nine equations (iii).] Here, in any case, is a system of equations with a preferred vector field u^a—a system, therefore, to which the present methods can be applied. Thus, we take the derivative system, as described in Appendix A, and then the Lagrange formulation, as described in Sec. II. There results a Lagrange formulation for a dissipative, relativistic fluid.

There exist still other types of material systems, e.g., some that are not fluids at all. Consider, for example, the elastic solid. In one treatment[22] of such a system in relativity, the fields consist of a unit timelike vector field u^a (the material 4-velocity), a positive function ρ (the mass density of the material), and a symmetric tensor field h_{ab} satisfying $h_{ab} u^b = 0$. This h_{ab} represents the geometry of the material as it was ''frozen in'' at the time the material originally solidified: It describes the shape to which the material would ''like to return.'' Thus, the combination $h_{ab} - (g_{ab} + u_a u_b)$, the difference between this natural geometry and the actual spatial geometry in which the material currently finds itself, is interpreted as the strain of the solid material. The equations on these fields are $\mathcal{L}_u h_{ab} = 0$ (the vanishing of the Lie derivative of h_{ab}, interpreted as asserting that the material remembers, over time, its frozen-in geometry), and $\nabla_b (\rho u^a u^b + \tau^{ab}) = 0$, (interpreted as the conservation of stress-energy, whence τ^{ab} is interpreted as the stress of the material). Here, τ^{ab} is to be given as some fixed function of h_{ab}, g_{ab}, and u^a. This is the stress–strain relation. Provided this stress–strain relation is chosen appropriately, the final system, it turns out, has an initial-value formulation: Specifically, it has a hyperbolization and no constraints. Again, we have a system to which the present methods can be applied. There results a Lagrange formulation for an elastic solid.

There are, presumably, a variety of other systems of equations, representing ''materials'' of various sorts, having, among their fields, a preferred 4-velocity. Examples might include the systems for a plasma, for a superconductor, or for a solid (such as ice) that is able to flow. These systems, too, will have Lagrange formulations.

These various material systems may, of course, interact with their environment in a variety of ways, e.g., electromagnetically, gravitationally, or through contact forces. What impact do such interactions have on their Lagrange formulations?

Consider, as an example, the fluid of Eqs. (20)–(21) interacting electromagnetically. This charged-fluid system is described by fields consisting of the original fluid variables, u^a and φ, together with an antisymmetric (electromagnetic) tensor field F_{ab}. The equations on these fields consist of Eq. (20), modified by the inclusion of a term on the right of the form $\mu F^b{}_a u^a$, Eq. (21)[23] and Maxwell's equations,

$$\nabla^b F_{ab} = \sigma u_a , \tag{26}$$

$$\nabla_{[a} F_{bc]} = 0. \tag{27}$$

Here, the μ in the first equation and the σ in Eq. (26) must be given as fixed fields on the manifold S of fluid states. The field σ describes how the fluid drives the electromagnetic field, and so is interpreted as the charge density. We require that it satisfy charge conservation: $V^\alpha \nabla_\alpha \sigma = \sigma$, $T^\alpha \nabla_\alpha \sigma = 0$. The field μ, which describes how the electromagnetic field drives the fluid, might be called the specific charge density. [For a normal fluid, σ and μ are in ratio $(\rho + p)$.] Here, in any case, is a list of fields, together with a system of equations on those fields. This system has an initial-value formulation, which it inherits from the separate initial-value formulations for the original fluid system [(20)–(21)] and for Maxwell's equations. We wish to take the Lagrange formulation for this system. Since the system does not express the derivative of u^a in terms of the other fields, the first step is to take the derivative system. But note that, in taking the derivative system, it is necessary to introduce, not only the new fields $w_a{}^b$ and $\zeta_a{}^\alpha$ that represent [via Eqs. (22) and (24), respectively] the derivatives of u^b and φ, but also the field ζ_{abc} that represents [via Eq. (A5)] the derivative of F_{ab}. One might have hoped that it would be possible, exploiting somehow the fact that our system of equations splits naturally into ''fluid equations'' and ''Maxwell-field equations,'' to avoid introducing the additional field ζ_{abc}. Unfortunately, this seems not to be the case. This issue is discussed briefly in Appendix A. In any case, this derivative system has the appropriate form (a preferred vector field u^a, whose derivative is expressed in terms of the fields of the system), and an initial-value formulation (which it inherits from that of the original coupled system). So, we may apply the methods of Sec. II. Thus, there is a Lagrange formulation for a charged fluid, but it requires the introduction of a further field ζ_{abc}, representing the derivative of the Maxwell field.

In a similar way, we may write down the Lagrange formulation for a charged dissipative fluid, a charged elastic solid, etc. In each of these cases, it is necessary to introduce the auxiliary field ζ_{abc}.

The situation for gravitational interactions is similar. Consider, again, the fluid of (20)–(21), now interacting gravitationally. The interacting system is described by fields consisting of the original fluid variables, u^a and φ, together with the variables for gravitation: a Lorentz-signature metric g_{ab}, and a derivative operator, ∇_a. The equations of this system consist of Eqs. (20)–(21),[24] the equation $\nabla_a g_{bc} = 0$, and Einstein's equation,

$$G_{ab} = T_{ab} , \tag{28}$$

where G_{ab} is the Einstein tensor. Here, T_{ab} is some fixed symmetric tensor function of g_{ab} and the fluid variables (which we interpret as the stress-energy tensor of the fluid). It plays a role analogous to that of the functions μ and σ for electromagnetic interactions. We demand of this tensor function that, as a consequence of Eqs. (20)–(21), it be conserved.[25] This system of equations does *not* have an initial-value formulation, in the sense we are using this term. But this is merely a consequence of the fact that our sense of this term is overly restrictive, in that it does not tolerate the diffeomorphism freedom characteristic of all systems in general relativity. In a physical sense, i.e., once the diffeomorphism freedom has been treated properly, the fluid-Einstein system does, of course, have an initial-value formulation. Now take the derivative system of this system. Note that in doing so we must, as in the electromagnetic case, include also fields to represent the derivatives of the gravitational fields.[26] Take the Lagrange formulation of the result. The resulting system, again, will not have an initial-value formulation in our restrictive sense, but it will have such a formulation when the diffeomorphism-freedom is properly taken into account. We conclude, then, that there does exist a Lagrange formulation for a gravitating fluid, but that it requires that we introduce further fields to represent the derivatives of the gravitational fields.

In a similar way, we may write down the Lagrange formulation for a gravitating dissipative fluid, a gravitating elastic solid, etc. In each case, it is necessary to introduce fields representing the derivatives of the gravitational fields; and in each case the Lagrange formulation retains the initial-value formulation of the original system.

A similar treatment is available for systems consisting of two or more different materials in interaction. In these cases, there will be two or more 4-velocity fields present, and we shall have to select one to be that with respect to which the Lagrange formulation is taken.

The treatment of systems in which several interactions are turned on simultaneously, e.g., the charged gravitating fluid, is similar.

Finally, we briefly characterize, within the present framework, Friedrich's[17] original example of a relativistic Lagrange formulation. Begin with the system for a gravitating fluid, as described above, for the case in which the fluid has a 2-dimensional manifold S of local states, i.e., that in which $T^{\alpha}=0$ and $V^{\alpha}=(\rho+p)\partial/\partial\rho|_n+n\partial/\partial n|_\rho$. For this system, first take the derivative system, and then the Lagrange formulation. The result of this process—after three, essentially cosmetic, further modifications—is precisely Friedrich's original example. The three further modifications are the following.

(1) Introduce, already in the original Einstein-fluid system, before taking the derivative system, a 3-dimensional space of additional variables, consisting of three unit vector fields, x^a, y^a, and z^a, that are required to be orthogonal to each other and to the 4-velocity u^a. On these fields, impose the equations that they be Fermi-transported by u^a. The introduction of these fields with these equations does not interfere with the initial-value formulation. These fields, which have no direct physical significance, are introduced to facilitate the writing of various equations.

(2) After taking the derivative system, but before passing to the Lagrange formulation, suppress half of the field ${\zeta_a}^{\alpha}$, which represents the derivative of the fluid state.[27] While such suppression of variables will in general destroy the initial-value formulation for a system, it turns out that, in this particular instance, it does not. Thus, the essential effect of this modification is to reduce by four the number of independent variables.

(3) Write the final equations, after passing to the Lagrange formulation, not in terms of the specific fields listed above, but rather in terms of others that are algebraic functions of these. This choice of variables—choice of ''coordinates'' on the bundle space—is, of course, a matter of convenience.

IV. CONCLUSION

We have introduced a scheme that takes a first-order, quasilinear system of partial differential equations and produces from it a new first-order, quasilinear system, its ''Lagrange formulation.'' The key requirement, on a given system of equations, in order that this scheme be applicable to it is that that system have, among its fields, some nowhere-vanishing vector field. Why this special role of a vector field? Could, for example, a similar scheme be developed based on some other geometrical object(s)? It turns out that there are two special features of vector fields that we used in the construction of the Lagrange formulation.

First, nowhere-vanishing vector fields on manifolds are locally homogeneous. This means the following. Let there be given any manifold M, any nowhere-vanishing vector field u^a thereon, and any point $x\in M$; and, similarly, some other manifold $\hat{M}$ (of the same dimension), vector field $\hat{u}^{\hat{a}}$ and point $\hat{x}\in\hat{M}$. Then there always exists a diffeomorphism between neighborhoods of x and $\hat{x}$

that sends u^a to $\hat{u}^{\hat{a}}$. In other words, nowhere-vanishing vector fields are ''locally all the same:'' They carry no local structure. We used this fact in Sec. II in order to replace u^a on M by some kinematical field $\hat{u}^{\hat{a}}$ on $\hat{M}$.

Second, by virtue of the appearance of the vector field $\hat{u}^{\hat{a}}$ on the left in Eq. (18), the system (6), (18) for the two-point tensor $\kappa_{\hat{a}}{}^{b}$ admits a hyperbolization. We used this fact in Sec. II in order to achieve a hyperbolization, and consequently an initial-value formulation, for the entire system (5)–(6), (18)–(19).

It appears that, given any other geometrical structure manifesting these two features, then there could be developed a ''Lagrange formulation'' based on it. It is only necessary to make three key modifications in Sec. II (all involving replacing the vector field by the totality of fields in the new geometrical structure): (i) Replace Eq. (10) by equations for the derivatives of all the fields of the geometrical structure; (ii) endow the base manifold $\hat{M}$ of the Lagrange formulation with kinematical fields consisting of all the fields of the geometrical structure; and (iii) replace Eq. (18) by the corresponding equation involving all the fields of the geometrical structure. Unfortunately, it is not so easy to find geometrical structures having the two features described above, in part because they are somewhat in opposition to each other: The first feature, local homogeneity, prefers fewer fields, relatively devoid of structure; while the second feature, hyperbolicity of (6), (18), prefers many fields, of rich structure.

There are a variety of geometrical structures that are locally homogeneous. Examples include: two commuting, pointwise independent vector fields; a nowhere-vanishing, curl-free 1-form; a symplectic structure; a flat, Lorentz-signature metric. Examples of geometrical structures that yield a hyperbolization for (6), (18) are somewhat less plentiful. One simple class consists of those in which the geometrical structure is comprised of a nowhere-vanishing vector field u^a, together with any additional fields of whatever type. For structures in this class, a hyperbolization for (6), (18) (suitably generalized) is guaranteed already by the presence of the vector field u^a in the structure.

Here is an application of these ideas. Consider the geometrical structure consisting of a nowhere-vanishing vector field u^a, together with a nowhere-vanishing 3-form, ω_{abc}, that has zero curl and is annihilated by u^a. This structure satisfies both of the features above—it is locally homogeneous, and it gives rise to a hyperbolization for (6), (18). So, this geometrical structure could serve as the basis for a Lagrange formulation. In fact, this formulation is appropriate for a physical system, namely that of a fluid with a 2-dimensional manifold of internal states, as discussed in Sec. III. Identify u^a with the velocity field of the fluid, and ω_{abc} with the particle-number density, via $\omega_{abc}=n\epsilon_{abcd}u^d$.

It is curious that the original system and its Lagrange formulation, while so similar with regard to their solutions, are completely different with regard to their initial-value formulations. Indeed, as we have seen in Sec. II, it is frequently the case that the original system of equations (4) has an initial-value formulation, while its Lagrange formulation, (5)–(7), does not. Perhaps there is some more natural or more general notion of ''initial-value formulation'' that would resolve this disparity.

APPENDIX A: DERIVATIVE SYSTEMS

Fix, once and for all, a first-order, quasilinear system of partial differential equations, as described in Sec. II. That is, fix a fiber bundle, with bundle manifold $\mathcal{B}$, base manifold M, and projection mapping $\mathcal{B}\xrightarrow{\pi}M$, together with smooth fields $k^{Aa}{}_{\alpha}, j^A$ on the bundle manifold $\mathcal{B}$. Our system of equations, on a cross-section, $M\xrightarrow{\phi}\mathcal{B}$, of this fiber bundle, is given by

$$k^{Aa}{}_{\alpha}(\nabla\phi)_a{}^{\alpha}=j^A. \tag{A1}$$

We shall now construct from this system a new first-order, quasilinear system of partial differential equations. The idea is to ''take one derivative'' (with respect to the point of M) of Eq. (A1).

The first step is to introduce the appropriate bundle of fields for the new system. Let the base manifold again be M. But now let the fiber, over a point $x \in M$, consist of all pairs, $(\phi, \zeta_a{}^\alpha)$, where ϕ is point of $\mathcal{B}$ satisfying $\pi(\phi) = x$ and $\zeta_a{}^\alpha$ is a tensor at ϕ satisfying

$$k^{Aa}{}_\alpha \zeta_a{}^\alpha = j^A. \tag{A2}$$

Thus, ϕ is merely a point of the fiber over $x \in M$, in the original bundle $\mathcal{B}$. It represents a set of ''values for the original fields'' at x. The tensor $\zeta_a{}^\alpha$ represents a set of ''values for the derivatives of the original fields.'' In order that a given $\zeta_a{}^\alpha$ be a viable candidate for these derivatives, it must satisfy Eq. (A2), the algebraic equation that results from replacing $(\nabla \phi)_a{}^\alpha$ in Eq. (A1) by $\zeta_a{}^\alpha$. We impose this algebraic condition on ζ in the very construction of the new bundle (as opposed, e.g., to introducing it later as an ''algebraic constraint''). In short, the *dynamics* [Eq. (A1)] of the original system goes into the *kinematics* [Eq. (A2)] of the new system. Call the bundle space of this new fiber bundle $\mathcal{B}'$. Thus, the dimension of the fibers of $\mathcal{B}'$ is given by: (dim fibers of $\mathcal{B}$) $(1+\dim(M))$-(dim vector space of equations in $\mathcal{B}$).

Consider, as an example, Maxwell's equations. Then M is a 4-dimensional manifold, with fixed smooth metric g_{ab} of Lorentz signature. For the bundle $\mathcal{B}$, the fiber over $x \in M$ consists of all antisymmetric tensors, F_{bc}, at x. Equation (A1) is Maxwell's equations: $g^{ab}\nabla_a F_{bc} = 0$, $\nabla_{[a}F_{bc]} = 0$. For this example, the new bundle, $\mathcal{B}'$, has, as its fiber over $x \in M$, the collection of all pairs, (F_{bc}, ζ_{abc}), with symmetries $F_{bc} = F_{[bc]}, \zeta_{abc} = \zeta_{a[bc]}$, and with ζ satisfying the algebraic conditions [Eq. (A2)] $g^{ab}\zeta_{abc} = 0, \zeta_{[abc]} = 0$. Thus, the fibers of $\mathcal{B}$ have dimension six, those of $\mathcal{B}'$ dimension twenty-two.

Returning to the general case, the second step is to introduce appropriate equations on this bundle. A cross-section of the bundle $\mathcal{B}'$ consists of fields $\phi, \zeta_a{}^\alpha$ on M. On such a cross-section, we impose the following system of partial differential equations:

$$(\nabla \phi)_a{}^\alpha = \zeta_a{}^\alpha, \tag{A3}$$

$$\nabla_{[a}\zeta_{b]}{}^\alpha = f_{ab}{}^\alpha. \tag{A4}$$

Equation (A3) provides the ''interpretation'' of ζ, as the derivative of ϕ. The $f_{ab}{}^\alpha$ on the right of (A4) is some field on $\mathcal{B}'$ [i.e., some function of (x, ϕ, ζ)], whose exact form depends on what derivative operator is used on the left side of that equation. The general rule is that Eq. (A4) to be the result of taking the curl of Eq. (A3). For example, if ϕ is represented by tensor fields over M, if ζ is represented by the tensor fields obtained by taking the covariant derivatives (with respect to some fixed derivative operator on M) of those fields, and if that same derivative operator is used on the left in Eq. (A4), then f will consist of certain terms involving ϕ and the curvature tensor of that derivative operator. If, on the other hand, all bundles are taken as simple products, and all derivatives are taken using the corresponding (flat) connection, then $f_{ab}{}^\alpha = 0$. Note that we have *not* included in our system the derivative of Eq. (A2). The reason is that Eq. (A2) has already been included at the algebraic level in the construction of the bundle $\mathcal{B}'$. Its derivative is thus an identity in $\mathcal{B}'$. On the other hand, we *do* include in our system Eq. (A4), even though it merely results from taking a derivative of Eq. (A3). In short, all algebraic conditions on fields are included in the construction of the bundle,[28] while all differential conditions on fields are included in the equations on a cross-section of that bundle. We note that the system of Eqs. (A3)–(A4) is indeed first-order and quasilinear.

Consider again the example, above, of Maxwell's equations. Then a cross-section of bundle $\mathcal{B}'$ consists of smooth fields, F_{bc}, ζ_{abc}, satisfying everywhere the symmetries and algebraic conditions given above. The equations, (A3)–(A4), on such a cross-section become, respectively,

$$\nabla_a F_{bc} = \zeta_{abc}, \tag{A5}$$

$$\nabla_{[d}\zeta_{a]bc} = 2R_{da[b}{}^{m}F_{c]m}\,. \tag{A6}$$

Given a system, consisting of bundle $\mathcal{B}$ and partial differential equations (A1), then by its *derivative system* we mean the system, consisting of bundle $\mathcal{B}'$ and partial differential equations (A3)–(A4), constructed above. Note that every solution of the original system gives rise to a solution of its derivative system [by merely setting $\zeta_a{}^{\alpha} = (\nabla\phi)_a{}^{\alpha}$]. Conversely, every solution of the derivative system gives rise to a solution of the original system (by merely ignoring ζ). The two systems of partial differential equations are, in this sense, "equivalent as to solutions." But they are not "equivalent as to form," a feature we exploit in Sec. II.

We next turn to the issue of the existence of an initial-value formulation for these systems. As discussed in Appendix B, we say that a general first-order quasilinear system (A1) of partial differential equations admits an *initial-value formulation* provided it satisfies the following three conditions: (i) the system admits a hyperbolization; (ii) all constraints of the system are integrable; and (iii) the system has the correct number of equations relative to the number of its unknowns. See Appendix B for the details of what these conditions mean. A key property of the derivative system is the following: *If the original system, (A1) admits an initial-value formulation, then so does its derivative system, (A3)–(A4).* We check the three conditions in turn.

Let the original system (A1) admit a hyperbolization (say, $h_{\beta A}$, with w_a). Then, we claim, so does its derivative system. Indeed, the corresponding bilinear expression [on a pair of tangent vectors, represented as $(\delta\phi^{\alpha}, \delta\zeta_a{}^{\alpha})$ and $(\delta'\phi^{\alpha}, \delta'\zeta_a{}^{\alpha})$] is given by

$$w_m h_{\alpha A} k^{Am}{}_{\beta}[\overset{+}{g}{}^{ab}\,\delta\zeta_a{}^{\alpha}\,\delta'\zeta_b{}^{\beta} + \delta\phi^{\alpha}\,\delta'\phi^{\beta}], \tag{A7}$$

where $\overset{+}{g}{}^{ab}$ is any positive-definite metric field on M. It is apparently not known whether the converse is true, i.e., whether the existence of a hyperbolization for the derivative system, (A3)–(A4), implies the existence of a hyperbolization for the original system (A1). Simple examples suggest that this is a reasonable conjecture.

Integrable constraints of the system (A1) do *not* lead to constraints of the corresponding derivative system. Rather, they lead to a reduction in the number of effective equations. Indeed, let $c^b{}_A$ be any constraint. Then the result of contracting Eq. (A4) with $c^a{}_A k^{Ab}{}_{\alpha}$ is an identity: It holds automatically, by virtue of Eq. (A2). Thus, each constraint for the system (A1) reduces by one the number of effective equations represented by Eqs. (A3)–(A4).

What, then, *are* the constraints of the derivative system (A3)–(A4)? These fall into two classes. The first class consists of those constraints that correspond to taking the curl of Eq. (A3). These constraints are of course integrable: Their integrability conditions are precisely (A4). The second class of constraints consists of those that correspond to taking the curl of Eq. (A4). These constraints, too, are integrable, by virtue of the fact that Eq. (A4) is itself a curl. Not all of these constraints, it turns out, are in general algebraically independent.

Let us return to our original partial differential equation (A1). Denote by n the dimension of the base manifold M (the "number of independent variables"), by u the dimension of the fibers in the bundle $\mathcal{B}$ (the "number of unknown functions"), and by e the dimension of the vector space of equations, (A1). Further, denote by $\hat{c}$ the dimension of the vector space of constraints, and, for fixed nonzero covector w_a, by c the dimension of the space of vectors of the form $w_a c^a{}_A$, as $c^a{}_A$ runs over the constraints. Then, as discussed in Appendix B, the condition that the original system (A1) have the "correct number of equations" becomes $e - c = u$. We turn now to the derivative system (A3)–(A4). The number of its unknowns is given by $u' = u + (nu - e)$ (the two terms representing the numbers of unknowns contained in the fields ϕ and ζ, respectively). The number of its equations is given by $e' = nu + [un(n-1)/2 - \hat{c}]$ [the two terms representing the number of effective equations in (A3) and (A4)], respectively. Finally, the number of effective constraints of the derivative system is given by $c' = u(n-1) + [(n-1)(n-2)u/2 + c - \hat{c}]$, (the two terms representing the number of effective constraints in (A3) and (A4), respectively[29]). From these formulas, it is easy to check: If $e-c=u$, then $e'-c'=u'$. In other words, if the original system has the correct number of equations, then so does the derivative system.

We conclude, then, that, beginning with a system (A1) having an initial-value formulation, its derivative system, (A3)–(A4), also has an initial-value formulation.

The construction above of the derivative system is useful because it permits a large class of systems of partial differential equations to be cast into a form to which the Lagrange formulation of Sec. II can be applied. But, unfortunately, passing to the derivative system and then to the Lagrange formulation is often a cumbersome procedure. The reason is that the derivative system requires the introduction of additional fields to represent the derivatives of *all* the fields of the original system—even of those fields only remotely related to the one real interest: the velocity field. The result is a large number of extraneous fields. More useful would be a construction that goes only part way to the full derivative system—one that introduces additional fields to represent the derivatives of only *some* of the original fields, leaving the remaining ones intact. It turns out that, while there are one or two systems (e.g., that for dust) for which a smaller derivative system along these lines is available, for the vast majority of systems of partial differential equations of physical interest there is none. Here, briefly, is why.

First, we must designate which of the dependent variables (the fields represented by ϕ) are to be derived and which not. This is done by writing the original bundle, $\mathcal{B}$, as a product of two bundles, $\mathcal{B}'$ and $\mathcal{B}''$, with the same base space[30] M. The bundle $\mathcal{B}'$ carries the fields whose derivatives will be represented by new variables, while $\mathcal{B}''$ carries the remaining fields. A cross-section ϕ of $\mathcal{B}$ consists precisely of a pair, (ϕ', ϕ''), where ϕ' is a cross-section of the bundle $\mathcal{B}'$, and ϕ'' is a cross-section of the bundle $\mathcal{B}''$. In terms of these variables, Eq. (A1) becomes

$$k'^{Aa}{}_{\alpha'}(\nabla\phi')_a{}^{\alpha'}+k''^{Aa}{}_{\alpha''}(\nabla\phi'')_a{}^{\alpha''}=j^A, \tag{A8}$$

where primed Greek indices denote tensors in $\mathcal{B}'$, and double-primed in $\mathcal{B}''$. Here, the fields k', k'' and j are all functions on $\mathcal{B}$, i.e., are functions of (x,ϕ',ϕ''). We now proceed just as with the derivative system. Introduce a new fiber bundle, with base manifold again M, but with fiber over $x\in M$ consisting of certain triples, $(\phi',\zeta_a{}^{\alpha'},\phi'')$. There must now be imposed on such triples all those algebraic conditions that flow from (A8). This is done as follows. At each point, denote by V the vector space of μ_A satisfying $\mu_A k''^{Aa}{}_{\alpha''}=0$. That is, V captures "those equations in (A8) that contain no derivative of ϕ''." We now demand, in order that a triple $(\phi',\zeta_a{}^{\alpha'},\phi'')$ give rise to a point of the fiber, the following: For every $\mu_A\in V$, $\mu_A k'^{Aa}{}_{\alpha'}\zeta_a{}^{\alpha'}=\mu_A j^A$. This is the fiber bundle for our new system. Let the equations of the new system be

$$(\nabla\phi)_a{}^{\alpha'}=\zeta_a{}^{\alpha'}, \tag{A9}$$

$$\nabla_{[a}\zeta_{b]}{}^{\alpha'}=f_{ab}{}^{\alpha'}, \tag{A10}$$

$$\nu_A k'^{Aa}{}_{\alpha'}\zeta_a{}^{\alpha'}+\nu_A k''^{Aa}{}_{\alpha''}(\nabla\phi'')_a{}^{\alpha''}=\nu_A j^A. \tag{A11}$$

In (A11), ν_A is any vector in some fixed subspace complementary to the subspace V. In other words, Eq. (A11) reflects those equations of (A8) that do involve the derivative of ϕ''.

The system (A9)–(A11) is, certainly, a first-order, quasilinear system of partial differential equations; and it has as its variables precisely the ones we intended, namely $(\phi',\zeta_a{}^{\alpha'},\phi'')$. But, unfortunately, this system is subject to a variety of maladies—and these can arise even if the original system was quite well-behaved. For example—and this happens frequently—there can be constraints for the system (A9)–(A11) that are hidden in Eq. (A11), and thus do not arise from any constraints for the original system, (A8). Furthermore, these new constraints are not in general integrable. One could attempt to include the integrability conditions of these new constraints as new equations for the system. But two further problems can arise. First, some integrability conditions can turn out to be mere algebraic equations on the fields, $(\phi', \zeta_a{}^{\alpha'}, \phi'')$. The only way to ''include'' such equations is to start over, introducing a new bundle right from the beginning. Second, some integrability conditions can turn out to be quadratic, rather than linear, in the field-derivatives. These cannot simply be ''included''—at least, not if we wish to retain a quasilinear system. The system (A9)–(A11) can also manifest a number of other types of difficulties, e.g., the absence of a hyperbolization or the wrong number of equations. There appears to be no simple, general condition that guarantees that Eqs. (A9)–(A11) lead to a system with an initial-value formulation.

As an example of this construction consider again the simple fluid, (1)–(2). Let $\mathcal{B}'$ be the bundle whose fiber consists only of the variable u^a; and $\mathcal{B}''$ the bundle whose fiber consists only of the variable ρ. In this example, the vector space V, capturing those equations in (1)–(2) involving no derivative of ρ, is zero-dimensional. The corresponding new bundle space, then, is that whose fiber, over $x \in M$, consists of $(u^a, w_b{}^a, \rho)$, with u^a unit timelike and $w_b{}^a$ satisfying $g_{ac}u^c w_b{}^a = 0$ (unit-ness of u^a). The equations for the new system, in this example, are

$$\nabla_b u^a = w_b{}^a, \tag{A12}$$

$$\nabla_{[a} w_{b]}{}^c = R_{abd}{}^c u^d, \tag{A13}$$

$$(g^{am} + u^a u^m)\nabla_m p + (\rho + p) u^m w_m{}^a = 0, \tag{A14}$$

$$u^m \nabla_m \rho + (\rho + p) w_m{}^m = 0. \tag{A15}$$

This system has a new constraint [obtained by combining Eqs. (A14) and (A15) to obtain an expression for $\nabla_m \rho$, and then taking its curl], which turns out not to be integrable. But its integrability condition turns out to be quasilinear in field-derivatives, and so may be included as a further equation of the system. The resulting system in this case (but not for the case of an even slightly more complicated fluid) actually admits a hyperbolization.

APPENDIX B: INITIAL-VALUE FORMULATION

Consider a first-order, quasilinear system of partial differential equations, as described in Sec. II. That is, we have a fiber bundle, with base manifold M, bundle manifold $\mathcal{B}$, and projection mapping $\mathcal{B} \xrightarrow{\pi} M$. The system of partial differential equations, on a cross-section, $M \xrightarrow{\phi} \mathcal{B}$, of this bundle, is given by Eq. (3). We are concerned here with the issue of under what circumstances such a system admits an initial-value formulation, i.e., a formulation in which the fields are first specified on some ''initial surface'' in M, and are then determined elsewhere in M by Eq. (3) itself.

The key to achieving such a formulation is an object called a *hyperbolization* of the system (3), a field $h_{\beta A}$ on the bundle manifold $\mathcal{B}$ having the properties described below. Consider, for (x, ϕ) any point of the bundle manifold $\mathcal{B}$, w_m any covector at $x \in M$, and $\delta\phi^\alpha, \delta'\phi^\alpha$ any two vectors at $(x,\phi) \in \mathcal{B}$ tangent to the fiber (''vertical''), the expression

$$w_m h_{\beta A} k^{Am}{}_\alpha \delta\phi^\alpha \delta' \phi^\beta. \tag{B1}$$

We demand, in order that this $h_{\beta A}$ be a hyperbolization, that, everywhere in $\mathcal{B}$, this expression be symmetric in $\delta\phi^\alpha, \delta'\phi^\alpha$ for all w_m, and positive-definite (i.e., positive for any nonzero $\delta'\phi^\beta = \delta\phi^\beta$) for some w_m. The most direct way to specify a hyperbolization for a system of partial differential equations is simply to give the bilinear expression (B1). Such an expression indeed defines a hyperbolization provided it is symmetric and positive-definite, as described above, and furthermore, that it is some multiple of the result of replacing, in the left side of Eq. (3), ''$(\nabla\phi)_a{}^\alpha$'' by ''$w_a\, \delta\phi^\alpha$.'' As an example, consider the system, (1)–(2), for a simple perfect fluid. Consider the bilinear expression

$$\delta' u^a[(\rho+p)(u^m w_m)g_{ab}\delta u^b+(\partial p/\partial\rho)w_a\delta\rho]$$
$$+(\partial p/\partial\rho)(\rho+p)^{-1}\delta'\rho[(\rho+p)\,\delta u^m\,w_m+u^m w_m\,\delta\rho]. \tag{B2}$$

We note that this expression is symmetric under interchange of the two vectors $(\delta\rho,\delta u^a)$ and $(\delta'\rho,\delta' u^a)$, and that [provided $(\rho+p)>0$ and $1\geqslant(\partial p/\partial\rho)>0$] it is positive-definite whenever w_m is future-directed timelike. Furthermore, this expression arises, as described above, from Eqs. (1)–(2). This bilinear expression, then, specifies a hyperbolization for this system.

Let there be given a hyperbolization, $h_{\alpha A}$, for the system (3). Then this object gives rise to an initial-value formulation for a portion of that system, in the following manner. Fix initial data, consisting of a submanifold T of M of codimension one (an ''initial surface'') together with a cross-section ϕ_0 over this submanifold (''data'' on that surface), such that at each point of T, the normal to T is one of the vectors w_m for which the bilinear expression (B1) is positive-definite (the surface is ''noncharacteristic''). Then, in some neighborhood of the submanifold T, there exists one and only one solution ϕ of the system

$$h_{\beta A}k^{Aa}{}_{\alpha}(\nabla\phi)_a{}^{\alpha}=h_{\beta A}j^A, \tag{B3}$$

such that $\phi=\phi_0$ on T. Note that we do not guarantee a solution of the entire system (3), but rather only of those components that are involved in the hyperbolization. While the proof of this theorem is technically difficult, the key idea is to construct, using the hyperbolization, an energy integral, which is positive-definite, and, effectively, conserved.

Denote by u the number of unknowns of the system (3) (i.e., the dimension of the fibers in $\mathcal{B}$), and by e the number of equations (i.e., the dimension of the vector space in which the index ''A'' lies). Then the mere existence of a hyperbolization for this system already implies $e\geqslant u$ (i.e., that there are at least as many equations as unknowns). Should it happen that this inequality is an equality, i.e., that $e=u$, then it follows that the hyperbolization tensor $h_{\alpha A}$ is invertible, and so that the system (B3) exhausts the original system of equations (3). Thus, in this case we are done: We have achieved our full initial-value formulation. In the example of the simple perfect fluid above, for instance, we have $e=u=4$, and so the hyperbolization (B2) gives rise to an initial-value formulation for the fluid system (1)–(2). Unfortunately, in many cases of interest we have the strict inequality $e>u$, i.e., there are additional equations in (3) that are not accounted for in (B3). Such ''additional equations'' are dealt with in the following manner.

By a *constraint* of the system, (3), of partial differential equations, at a point of $\mathcal{B}$, we mean a tensor $c^a{}_A$ at that point such that the tensor $c^a{}_A k^{Ab}{}_{\alpha}$ is antisymmetric in the indices ''a,b.'' This definition has two facets. First, each constraint gives rise to an integrability condition. Fix a constraint field, $c^a{}_A$, and a solution ϕ of Eq. (3). Contract both sides of Eq. (3) with $c^b{}_A$, and apply to both sides some derivative operator, ∇_b, on M. Then, by the constraint-condition, terms involving second derivatives of ϕ vanish, leaving an algebraic equation (indeed, a polynomial of degree at most two) in the first derivative, $(\nabla\phi)_a{}^{\alpha}$, of ϕ. The constraint field is said to be *integrable* if this equation is an algebraic consequence of Eq. (3), i.e., if the difference of its two sides is the product of some expression (at most linear in field-derivatives) and the difference of the two sides of (3). The lack of integrability of a constraint generally indicates that ''not all the equations have been included in the original system (3).'' As to the second facet, each constraint gives rise to a compatibility condition on initial data. Fix constraint field, $c^a{}_A$, solution ϕ of Eq. (3), and submanifold T of M of codimension one. Then, at each point of T, we have

$$n_m c^m{}_A k^{Aa}{}_{\alpha}(\nabla\phi)_a{}^{\alpha}=n_m c^m{}_A j^A, \tag{B4}$$

where n_m is the normal to T at that point. But, by virtue of the constraint-condition, the index ''a'' in the tensor $n_m c^m{}_A k^{Aa}{}_{\alpha}$ is tangent to T. Thus, Eq. (B4) takes the derivative of ϕ only in directions tangent to T, and so it refers only to the value of ϕ on T, i.e., only to the initial data on T. In short, Eq. (B4) represents a compatibility condition on initial data. If these compatibility conditions were not satisfied, then we would have have no hope of finding a corresponding solution of Eq. (3). As an example, consider the Maxwell equation $\nabla_{[a}F_{bc]}=0$. This equation has a constraint. The corresponding integrability condition, obtained by taking the curl of this equation, is an identity, and so this constraint is integrable. The compatibility condition (B4) on initial data becomes, in this example, $\nabla\cdot B=0$.

In the case in which $e>u$, i.e., in which the system (3) has more equations than unknowns, two further conditions must be imposed on the system. The first is that all the constraints be integrable. The second is that $e-c=u$, where c denotes the dimension of the vector space of vectors of the form $w_m c^m{}_A$, for fixed w_m, as $c^m{}_A$ runs through all the constraints. This last condition means that any additional equations in (3) that are not included already in (B3) are accounted for, effectively, by constraints. It states that (3) has the ''correct number of equations'' for its unknowns. In the case of Maxwell's equations, for example, all the constraints are integrable, as we have already remarked; and we have $e=8$, $c=2$, and $u=6$, so there is indeed the correct number of equations. That is, the two further conditions above are satisfied in this example.

Consider now a first-order, quasilinear system of partial differential equations that satisfies the three conditions given above. That is, let the system (i) admit a hyperbolization, (ii) have all its constraints integrable, and (iii) have the correct number of equations, as described above. It seems likely that such a system—possibly with some mild further conditions—must always manifest an initial-value formulation in some suitable sense. That is, we would expect that, given initial data for the system on a suitable surface T, satisfying on T the compatibility conditions (B4), then there exists a unique corresponding solution of Eq. (3) in a neighborhood of T. A key piece of evidence prompting this expectation is the following. There certainly exists a solution of Eq. (B3) manifesting the initial data, as we have already seen. Consider next the left sides of Eq. (B4) (as $c^m{}_A$ varies over all constraints). These expressions of course vanish on T, and, by virtue of the conditions (ii) and (iii) above, satisfy a system of equations that express the ''time-derivatives'' (off T) of these expressions in terms of their ''space-derivatives'' (within T). Naively, we might expect that, as a consequence, these expressions must vanish in a neighborhood of T. But the vanishing of these expressions implies, again by condition (iii) above, that Eq. (3) itself is satisfied everywhere in a neighborhood of T. Indeed, in all physical examples of which we are aware—including all those discussed in this paper—this naive expectation is in fact borne out. Unfortunately, there is, apparently, no general theorem to this effect. Nevertheless, we shall, for convenience, use the expression ''having an initial-value formulation'' to describe systems of partial differential equations that satisfy the three conditions, (i)–(iii), above.

[1] See, for example, R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves* (Interscience, New York, 1948), for the Euler and Lagrange formulations of non-relativistic perfect fluids, and Appendix A of Ref. 2 for the Euler formulation of a relativistic perfect fluid.

[2] R. Geroch, ''Partial differential equations of physics,'' in *General Relativity*, Proceedings of the 46th Scottish Universities Summer School in Physics, edited by G. S. Hall and J. R. Pulham (SUSSP Publications, Edinburgh; IOP, London, 1996). Available as gr-qc/9602055.

[3] For the case of the Einstein-Euler system, for example, see Sec. 4.2 of Ref. 4, and references therein.

[4] H. Friedrich and A. Rendall, ''The Cauchy problem for the Einstein equations,'' in *Einstein's Field Equations and their Physical Interpretation*, edited by B. G. Schmidt (Springer-Verlag, Berlin, 2000), available as gr-qc/0002074.

[5] In fact, some care must be taken, in the Lagrange formulation, even to say what ''initial-value formulation'' means, in light of the fact that the independent variables are not the usual space-time events, through which evolution normally proceeds.

[6] Recall that this condition requires, essentially, that, locally in M, $\mathcal{B}$ can be written as a product, $M\times F$, of M with some other fixed manifold F, in such a way that the projection mapping π becomes the projection to the M-factor in this product. This condition guarantees, e.g., that, locally, all the fibres of the bundle are diffeomorphic with this fixed manifold F, and so with each other.

[7] Note that we can, in this case, convert these to ordinary tensors on the manifold $\hat{M}$ by using $\kappa_{\hat{a}}{}^b$ and its universe. This, a mere ''coordinate transformation'' on the fibres, changes nothing, in particular, not the final partial differential equations of the Lagrange formulation.

[8] For convenience, we shall always include within our system *all* first-order equations on the fields of the system, even those that arise from differentiating other equations of the system.

[9] These derivatives may be characterized in the following manner. Consider the bundle with base space $\hat{M}$ and fibre over $\hat{x}\in\hat{M}$ consisting of a pair, $(x,\kappa_{\hat{a}}{}^b)$, where $x\in M$ and $\kappa_{\hat{a}}{}^b$ is a tensor with indices at x and $\hat{x}$. Then a choice of connection in this bundle gives rise to an operator $\boldsymbol{\nabla}_{\hat{a}}$ for use in the left side of Eq. (6).

[10] A Lorentz metric on M, for example, becomes, on $\hat{M}$, an algebraic function of the fields (namely, just of x) of the Lagrange formulation.

[11] In the notation of (5)–(7), we have $\varphi=(x,\rho)$, and $\hat{\varphi}=(x,\hat{\rho})$.

[12] There is an unfortunate complication here, involving the normalization condition, $u^a u^b g_{ab}=-1$, on u^a. It is awkward simply to carry this condition through the Lagrange formulation. But there are several other ways—none very elegant—to deal with it. Perhaps the simplest is to rewrite the fluid equations from the outset [by inserting, strategically, factors of $(u^a u^b g_{ab})$] in such a way that, while retaining their initial-value formulation, they no longer require this normalization condition. Then take the Lagrange formulation of these new equations.

[13] Note in particular that the original system, (10)–(11), always possesses the constraints arising from the curl of Eq. (10). Thus, if the constraints of this system are to be integrable, this curl-equation must have been included in the system (10)–(11).

[14] These are not to be confused with the indices for tensors on the bundle space, used extensively in Sec. II.

[15] Note that the last two terms on the left in Eq. (21) constitute the most general expression (involving u^a and φ) quasilinear in the derivative of u^a.

[16] For "if," suppose that $V^\alpha F_\alpha > 0$ everywhere on S. It follows that there exists a positive-definite metric field, $g_{\alpha\beta}$, on the manifold S such that $V^\alpha g_{\alpha\beta}=F_\beta$ everywhere. Choose one (e.g., the sum of $F_\alpha F_\beta/(F_\gamma V^\gamma)$ and a suitable positive semi-definite tensor $h_{\alpha\beta}$ that annihilates V^α) and consider the bilinear expression

$$-(w_m u^m)[g_{ab}\delta u^a \delta' u^b + g_{\alpha\beta}\delta\varphi^\alpha \delta'\varphi^\beta] - w_m F_\alpha[\delta u^m \delta'\varphi^\alpha + \delta' u^m \delta\varphi^\alpha].$$

This bilinear expression indeed arises, as described in Appendix B, from Eqs. (20)–(21), and is indeed positive-definite (for w_m sufficiently close to u_m). So, this bilinear expression gives rise to a hyperbolization. The converse is easy.

[17] H. Friedrich, "Evolution equations for gravitating ideal fluid bodies in general relativity," Phys Rev. D **57**, 2317–2322 (1998).

[18] I. Müller, "Zum Paradox der Wärmeleitungstheorie," Z. Phys. **198**, 329–335 (1967).

[19] R. Geroch and L. Lindblom, "Causal theories of dissipative relativistic fluids," Ann. Phys. (NY) **207**, 394–416 (1991).

[20] R. Geroch, "Relativistic theories of dissipative fluids," J. Math. Phys. **36**, 4226–4241 (1995).

[21] I. Müller and T. Ruggeri, *Extended Thermodynamics*, in Springer Tracts in Natural Philosophy, 2nd ed. (Springer, New York, 1998), Vol. 7.

[22] See, e.g., Ref. 2. For other treatments, as well as the local existence theory for solutions, see Y. Choquet-Bruhat and L. Lamoureux-Brousse, "Sur les équations de l'élasticité relativiste," C. R. Acad. Sci. Paris **276**, 1317–1320 (1973); and also G. Pichon, "Théorèmes d'existence pour les équations des milieux élastiques," J. Math. Pures Appl. **45**, 395–409 (1966). For a brief summary of this subject, see Ref. 4.

[23] There could also be included on the right side of this equation terms algebraic in the electromagnetic and other fields. Such terms would represent, e.g., an effect of the electromagnetic field on the rates of chemical reactions.

[24] Note that there are no expressions, algebraic in the gravitational fields, that could be introduced on the right in these equations. This is a reflection of "the equivalence principle."

[25] The most general candidate for such a stress-energy (i.e., the most general algebraic function of our fields, having the correct index-structure) is given by $T_{ab}=(\rho+p)u_a u_b + p g_{ab}$, where ρ,p are some functions on the manifold S of fluid states. When does there exist such a T^{ab} that, in addition, is conserved, $\nabla_b T^{ab}=0$, by virtue of the field equations (20)–(21)? It is not difficult to check that (assuming $V^\alpha F_\alpha>0$; and demanding $\rho+p>0$) a necessary and sufficient condition is that the fields F_α, V^α, and T^α on S satisfy the following three equations: $F_{[\alpha}\nabla_\beta F_{\gamma]}=0$, $T^\alpha K_\alpha=0$, and $\nabla_{[\alpha}(K_{\beta]}+F_{\beta]})=0$, where we have set $K_\alpha=(2V^\beta \nabla_{[\beta}F_{\alpha]}+F_\alpha)/(V^\gamma F_\gamma)$.

[26] In the resulting system, there will initially be two versions of "the derivative of the metric g_{ab}," one being the original derivative operator ∇_a, and the other arising (via g_{ab}) through passage to the derivative system. These two versions are then to be set equal to each other, via Eq. (A2). A similar phenomenon occurs, e.g., on taking the derivative system of the Klein-Gordon system.

[27] This "suppression" proceeds, in more detail, as follows. Choose on the 2-manifold S, a function s (which is interpreted in Ref. 17 as the entropy per particle) satisfying $V^\alpha \nabla_\alpha s=0$. Now delete the field $\zeta_a{}^\alpha$ everywhere, by replacing the component $\zeta_a{}^\alpha \nabla_\alpha s$ of $\zeta_a{}^\alpha$ by some new field f_a, and the remaining components of $\zeta_a{}^\alpha$ by $(\nabla\varphi)_a{}^\alpha$. To the resulting system add those further equations that are required for integrability of the constraints.

[28] In fact, there is, at this level of generality, a possible anomaly with the system (A3)–(A4). In some cases, further *algebraic* conditions on the fields can follow from Eq. (A4). In fact, this anomaly will never arise in systems of interest, because it is precluded by the requirement, which we shall impose shortly, that all constraints of the original system (A1), be integrable.

[29] The number of effective constraints of Eq. (A4) is the dimension of the vector space of tensors $\Lambda^{ab}{}_\alpha$ satisfying $\Lambda^{ab}{}_\alpha = \Lambda^{[ab]}{}_\alpha$ and $w_a \Lambda^{ab}{}_\alpha=0$ (namely, $(n-1)(n-2)u/2$), minus the dimension of the vector space of such tensors of the form $c^a{}_A k^{Ab}{}_\alpha$ for $c^a{}_A$ a constraint (namely, $\hat{c}-c$).

[30] Recall that the product of two bundles, with the same base space M, is the bundle, again with the base space M, whose fibre, over point $x\in M$, is given by the product of the fibres, over x, in the separate bundles.