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On the causality of a dilute gas as a dissipative relativistic fluid theory of divergence type

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Abstract. The dissipative relativistic fluid theories of divergence type are the simplest theories which are physically consistent and have a well posed—hyperbolic—initial value formulation, since they can be constructed from a single scalar function χ and a dissipation-source tensor I^{ab}, both of them functions of fluid variables. In this work we find the expression for this generating function for the case of a dilute gas using only the knowledge of an equilibrium fluid state, which is known from the kinetic theory of dilute gases. The generating function is obtained by imposing some conditions on the divergence theory, related to the symmetry and trace of the tensor of the fluxes. These conditions come naturally from kinetic theory, and are needed to correctly describe a dilute gas. We prove that in the neighbourhood of the equilibrium states, these divergence type equations for a dilute gas are causal for Boltzmann, Fermi or Bose equilibrium distribution functions.

1. Introduction

The earliest theories of relativistic dissipative fluids developed by Eckart [1] and Landau and Lifshitz [2] are now known to be physically unacceptable, because they fail to provide causal evolution equations and their equilibrium states are unstable, in the sense that small spatially bounded departures from equilibrium at one instant of time will diverge exponentially with time [3], at a rate incompatible with the observed behaviour of normal fluids. In those theories one of the basic assumptions was that the usual fluid variables (a four velocity and two thermodynamic variables) were the only dynamical fields. More recently attempts have been made to formulate acceptable dissipative relativistic fluid theories involving an extended set of thermodynamic variables. In these theories [4–6], the dynamical variables are the entire stress–energy tensor T^{ab} and the particle-number current N^a. A particular set of these theories, in which all the dynamical equations for the variables T^{ab} and N^a can be written as total divergence equations, is known as divergence type theories [4, 5, 7].

This particular set of theories has some physically nice properties. By construction, the dynamical equations of these theories are symmetric, and it is straightforward to determine the conditions under which the full nonlinear evolution equations are hyperbolic and causal. These dynamical equations are determined giving a single generating function χ, and a dissipation-source tensor I^{ab}. Besides their simplicity this type of equation has the extra advantage that by being of divergence form, discontinuous solutions (shocks) may be given

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mathematical meaning, which is very important since solutions to these equations generally evolve to form shocks.

To decide whether these theories are physically satisfactory it would be useful to see whether they are able to describe the simplest physical situation where this can be applied, namely a dilute gas nearby equilibrium states. This implies relating† the generating function χ and the dissipation-source tensor I^{ab} with microphysical quantities such as the distribution function f and the collision term C from kinetic theory. One way to do this was proposed by Liu *et al* [5]. They relate tensors in divergence type theory with the three first moments of the distribution function. This identification allows the generating function χ to be determined near equilibrium, in terms of its values in an equilibrium state $\chi|_{\mathrm{E}}$. However, if the distribution function in an equilibrium state $f|_{\mathrm{E}}$ is known from kinetic theory, the generating function in an equilibrium state $\chi|_{\mathrm{E}}$ can be calculated; so, it is possible to build the generating function χ near equilibrium up to the order needed to show causality in the no-equilibrium variables, with knowledge of the equilibrium distribution function $f|_{\mathrm{E}}$ only.

The purposes of this paper are essentially three: first, to determine the generating function near equilibrium, following the ideas in [5], with the only knowledge of the equilibrium distribution function of the dilute gas; second, to prove causality for Boltzmann, Bose and Fermi gases; and, third, to show that our results about causality for Boltzmann's gas agree with previous ones [8]. They also agree with previous results in Stewart [9]. Nevertheless those results were obtained in a different way from that presented here.

In section 2.1 we review the formulation of dissipative relativistic fluid theories of divergence type, by Geroch and Lindblom [4] and Pinnisi [7]. In section 2.2 we review basic concepts on kinetic theory. In section 3 we build a divergence type theory that describes a dilute gas. In section 4 we study the causal properties of these theories. In section 5 we discuss some results.

2. A brief review of two known theories

2.1. Dissipative fluid theories of divergence type

Following [4, 5, 7], we define a dissipative fluid theory of divergence type as a theory with the following three properties.

(i) The dynamical variables can be taken to be the particle-number current N^a, and the (symmetric) stress–energy tensor T^{ab}.

(ii) The dynamical equations are

$$\nabla_a N^a = 0 \tag{2.1}$$

$$\nabla_a T^{ab} = 0 \tag{2.2}$$

$$\nabla_a A^{abc} = I^{bc} \tag{2.3}$$

where the tensors A^{abc} (tensor of fluxes) and I^{bc} (dissipation-source tensor) are local functions of the dynamical variables N^a and T^{ab}, and are trace free and symmetric in the last two indices.

(iii) There exists an entropy current s^a (local function of N^a and T^{ab}) which, as a consequence of the dynamical equations, must satisfy

$$\nabla_a s^a = \sigma$$

where σ is some positive function of N^a and T^{ab}.

† Up to the order in no-equilibrium variables needed to build the corresponding system of linearized equations around an equilibrium state.

It can be seen [4,7] that any theory with these three properties is determined by specifying a single scalar generating function χ and the tensor I^{ab} as functions of a new set of dynamical variables $\zeta, \zeta^a, \zeta^{ab}$ (with the latter being trace free and symmetric). The dynamical equations for these variables are (2.1)–(2.3), with

$$N_a = \frac{\partial^2 \chi}{\partial\zeta\,\partial\zeta^a} \tag{2.4}$$

$$T_{ab} = \frac{\partial^2 \chi}{\partial\zeta^a\partial\zeta^b} \tag{2.5}$$

$$A_{abc} = \frac{\partial^2 \chi}{\partial\zeta^a\partial\zeta^{bc}} \tag{2.6}$$

while the entropy current is determined by

$$s^a = \frac{\partial\chi}{\partial\zeta_a} - \zeta N^a - \zeta_b T^{ab} - \zeta_{bc}A^{abc}$$

with the source given by

$$\sigma = -\zeta_{ab}I^{ab}.$$

It is helpful to introduce ζ^A to represent the collection of dynamical variables: $\zeta^A = (\zeta, \zeta^a, \zeta^{ab})$. In the same way we introduce I_A to represent the dissipation-source tensor: $I_A = (0, 0, I_{ab})$. Equations (2.1)–(2.3) can then be written in this notation as

$$M^a_{AB}\nabla_a\zeta^B = I_A \tag{2.7}$$

with

$$M^a_{AB} = \frac{\partial^3\chi}{\partial\zeta_a\partial\zeta^A\partial\zeta^B}.$$

The system of equations (2.7) is then automatically symmetric since matrix M^a_{AB} is symmetric in indices A, B due to the fact that partial derivatives commute. We say that a symmetric system is hyperbolic in an open set of fluid states if there exists a future-directed timelike ω^a (possible state dependent), such that $\omega_a M^a_{AB}$ in that neighbourhood is negative definite. A symmetric system is causal in an open set of fluid states if $\omega_a M^a_{AB}$ in that neighbourhood is negative definite for all future-directed timelike ω^a. The property of hyperbolicity ensures that system (2.7) has a well-posed initial-value formulation, while causality ensures that no fluid signals can propagate faster than light. It is natural for a relativistic dissipative fluid theory to demand causality; this implies conditions on the generating function χ.

Following Geroch and Lindblom [4], we define an equilibrium state as one on which the dynamics is time reversible. They conclude that in an equilibrium state of the theory the dissipation-source tensor I_{ab} and the dynamical field ζ^{ab} must vanish. They also conclude that at equilibrium $\nabla_a\zeta = 0$ and $\nabla_{(a}\zeta_{b)} = 0$ (that is, the variable ζ_a is a Killing vector). This concludes the review of the main results that we need from [4].

2.2. *Kinetic theory*

We consider a distribution of identical particles in the spacetime. The particles interact via short-range forces, idealized as point collisions, and via the gravitational field. A distribution function $f(x^a, p^a)$ is defined [6] by the statement that

$$f\frac{p^a}{m}\mathrm{d}\Sigma_a\,\mathrm{d}\omega$$

is the number of world-lines cutting an element of 3-surface $d\Sigma_a$ and having 4-momenta p^a which terminate on a cell of 3-area $d\omega$ on the mass shell $p_a P^a = -m^2$. The distribution function is the solution of the relativistic transport equation

$$p^a \left(\frac{\partial}{\partial x^a} + p_c \Gamma^c_{ab} \frac{\partial}{\partial p_b} \right) f(x^d, p^d) = C(x^d, p^d)$$

with C the collision term defined requiring that

$$C(x^d, p^d) \frac{d\omega}{m} \sqrt{-g} d^4 x$$

be the number of particles in the momentum range $d\omega$ around p_d which are created by collisions in the 4-volume $\sqrt{-g} d^4 x$, around the point of coordinates x^d.

The collision term C can be any arbitrary function provided that the resulting theory satisfies certain physical properties. The following general properties are usually required.

(i) The form of C is consistent with 4-momentum and particle number conservation at collisions.

(ii) The collision term C yields a non-negative expression for the entropy production.

To relate macroscopic quantities to the distribution function it is useful to introduce the n-moment associated with f, that is the following totally symmetric hierarchy of tensors

$$J^{a_1 \cdots a_n} \equiv \int p^{a_1} \cdots p^{a_n} f \, d\omega \qquad n = 0, \ldots, \infty.$$

By virtue of the transport equation, the $(n+1)$ moment satisfies

$$\nabla_a J^{a a_1 \cdots a_n} = I^{a_1 \cdots a_n} \tag{2.8}$$

with the source tensor $I^{a_1 \cdots a_n}$ defined as

$$I^{a_1 \cdots a_n} \equiv \int p^{a_1} \cdots p^{a_n} C \, d\omega.$$

Those moments are not independent quantities because they satisfy the following relations:

$$J^{a_1 \cdots a_j \cdots a_i \cdots a_n} g_{a_j a_i} = -m^2 J^{a_1 \cdots a_{n-2}} \qquad i, j = 1 \cdots n \qquad i \neq j. \tag{2.9}$$

We now describe a dilute gas. Following [10], this is done by building an appropriate collision term where elastic binary conditions give the dominant contribution. Assuming Boltzmann's ansatz, i.e. incoming particles are uncorrelated and this probability is proportional to the product $f(x^a, p^a) f(x^{a'}, p^{a'}) \, d\omega \, d\omega'$, Boltzmann's form of the collision term is given by

$$C = \int W(p^a, p^{a'}; p_1^a, p_1^{a'})(f(x^a, p_1^a) f(x^a, p_1^{a'}) - f(x^a, p^a) f(x^a, p^{a'})) \, d\omega' \, d\omega_1 \, d\omega_1'$$

where $W(p^a, p^{a'}; p_1^a, p_1^{a'})$ is the transition rate for two particles with incoming momenta p^a and $p^{a'}$ that are scattered with outgoing momenta p_1^a and $p_1^{a'}$. This transition probability is symmetric in $(p^a, p^{a'})$ and in $(p_1^a, p_1^{a'})$, and satisfies the microscopic reversibility: $W(p^a, p^{a'}; p_1^a, p_1^{a'}) = W(p_1^a, p_1^{a'}; p^a, p^{a'})$. Fermi or Bose statistics can be incorporated by introducing occupation probability factors $\Delta(x^a, p_1^a)$ and $\Delta(x^a, p_1^{a'})$ for the final states to allow Pauli exclusion or Bose–Einstein effects. The occupation probability factor is defined as

$$\Delta(x^a, p^a) = 1 + \varepsilon \frac{h^3}{\omega} f(x^a, p^a)$$

where h is Planck's constant, ω is the spin-weight (number of available states per quantum phase-cell) and ε is 1 for bosons and -1 for fermions.

By construction, this collision term satisfies property (i): defining the particle-number current and the stress–energy tensor respectively as the two first moments of the distribution function and using (2.8)

$$N^a \equiv J^a \qquad T^{ab} \equiv J^{ab}. \tag{2.10}$$

It also satisfies property (ii): defining an entropy current by

$$S^a(x^b) \equiv -\frac{1}{m}\int \phi p^a \,\mathrm{d}\omega \qquad \text{with} \qquad \phi = \left(f\ln(h^3 f) - \frac{\omega}{\epsilon h^3}\Delta\ln(\Delta)\right)$$

it can be checked that

$$\nabla_a S^a = -\frac{1}{m}\int y(f)\mathcal{C}(f)\,\mathrm{d}\omega \geqslant 0 \qquad \text{where} \qquad y(f) \equiv \phi'(f) = \ln\left(\frac{h^3 f}{\Delta}\right).$$

Now, the local equilibrium states are defined requiring that entropy production vanishes. This implies [6]

$$\mathcal{C}(f_0) = 0 \qquad \text{or} \qquad y(f_0) = y_0 = \frac{\alpha}{k} + \frac{1}{kT}u^a p_a.$$

It can be seen that one condition implies the other [11]. From the second one it is clear that this leads to a unique distribution function at equilibrium, namely

$$f_0(x^b, p^b) = \eta \bigg/ \left[\exp\left(-\frac{\alpha}{k} - \frac{1}{kT}u^a p_a\right) - \epsilon\right] \tag{2.11}$$

where T is the absolute temperature, α is a relativistic chemical potential per unit time $\alpha = [(\rho + p)/nT] - s$ (with ρ the energy density, p the pressure, s the entropy and n the particle-number density), k is Boltzmann's constant, $\eta = \omega/h^3$ with $\omega = (2\nu + 1)$ for particles with spin $\nu h/(2\pi)$ and ϵ is 1 for bosons and -1 for fermions, as above.

The distribution function for non-equilibrium states is unknown but it is possible to obtain information for small deviations from local equilibrium. The procedure is known as the Grad 14-moment approximation [6]. The main idea is to do a 14-parameter variation on $y(f)$ around its equilibrium value y_0, so it has the form

$$y(f) = \frac{\xi}{k} + \frac{1}{k}\xi_a p^a + \frac{1}{k}\xi_{ab} p^a p^b$$

where the 14-parameters are ξ, ξ_a, ξ_{ab}, functions of x^a, with ξ_{ab} trace free and symmetric†, and such that at equilibrium

$$\xi_0 = \alpha \qquad \xi_{0a} = \frac{u_a}{T} \qquad \xi_{0ab} = 0.$$

Then it can be checked [6] that it is possible to obtain a closed system of equations for the parameters of the perturbation, considering equations (2.8) only for the three first moments. So the system of equations can be written as

$$\nabla_a J^a = 0 \tag{2.12}$$

$$\nabla_a J^{ab} = 0 \tag{2.13}$$

$$\nabla_a J^{a\langle bc\rangle} = I^{\langle bc\rangle} \tag{2.14}$$

where the symbol $\langle\,\rangle$ means symmetrization and trace free.

We may summarize all this by stating that (2.12)–(2.14) are the equations for variables (ξ, ξ_a, ξ_{ab}) of a dilute gas near an equilibrium state.

† Because the trace variation in ξ_{ab} is equivalent to a ξ variation.

3. Dilute gas as a divergence type theory

In this section we will determine an expression for the generating function of a divergence type theory for states in the neighbourhood of equilibrium. This is done by imposing symmetry conditions on a certain tensor, as it is necessary to relate it with the third momentum tensor in kinetic theory. More specifically, recalling the assignation (2.10) for the first two moments of the distribution function, then the system of equations for the three first moments (2.12)–(2.14) could be thought as a divergence type theory (2.1)–(2.3). To do so, we must relate tensors J^{abc} and A^{abc}. Taking into account equations (2.4)–(2.6), this relation between tensors J^{abc} and A^{abc} imposes a condition on the generating function χ. To find this relation, we recall (2.9), that is

$$J^{ab}{}_b = -m^2 J^a = -m^2 N^a$$

and if we assume that

$$J^{ab} = T^{ab}$$

then the only possible relation between these tensors for the dynamical equation (2.3) to be equivalent to (2.14) is

$$J^{abc} = A^{abc} - \frac{m^2}{4} g^{bc} N^a. \tag{3.1}$$

Since J^{abc} is totally symmetric, this condition imposes conditions in A^{abc} and N^a (such that the right-hand side of (3.1) is totally symmetric), which, in turn, impose conditions on the generating function χ. These conditions are equivalent to the following system of equations for χ:

$$\frac{\partial^2\chi}{\partial\zeta^a\partial\zeta^{bc}} - \frac{\partial^2\chi}{\partial\zeta^b\partial\zeta^{ac}} - \frac{m^2}{4} g_{bc}\frac{\partial^2\chi}{\partial\zeta\partial\zeta^a} + \frac{m^2}{4} g_{ac}\frac{\partial^2\chi}{\partial\zeta\partial\zeta^b} = 0. \tag{3.2}$$

Where should these equations hold? One could argue that since the above identification $J^a \leftrightarrow N^a$, $J^{ab} \leftrightarrow T^{ab}$ is usually taken to hold only to first order off equilibrium, this should also be valid to that order. This requirement and the appropriate boundary conditions (the correct classical limit, as we show below) imply a unique generating function up to the order needed to study causality at equilibrium. One could try to push the identification further. Will this be possible? That is, will there exist exact solutions of (3.2); and if so, will they be essentially determined uniquely by the values of χ at equilibrium? If this were the case, then one would have dissipation theories essentially uniquely determined (as far as the principal part of the equations concerns) solely from knowledge of the generating function at equilibrium. This seems to be the case if one formally writes χ as a power series in terms of the dissipative variables and uses (3.2) to compute their coefficients in terms of the values of χ at equilibrium ($\zeta_{ab} = 0$). The constants of integration that appear in such a process can be eliminated by the condition of having an appropriate classical limit, so the determination of those coefficients in terms of values of χ at equilibrium is essentially unique. General results concerning the above questions would be published elsewhere. As will be explained below, for this work we are only interested in knowing the dynamical equations near an equilibrium state, so we will obtain the generating function χ by only considering terms up to the second derivative in dissipative variables.

The generating function χ is a scalar so it has to depend only on ζ, $\mu \equiv \sqrt{-\zeta^a\zeta_a}$ and scalars that can be constructed from ζ^{ab} (without loss of generality we can identify the variables $(\zeta, \zeta_a, \zeta_{ab})$ of the divergence type theories with the 14-parameters (ξ, ξ_a, ξ_{ab}) defined in the context of kinetic theory). We are interested in studying fluid states near equilibrium and, as we will see below, we will be concerned with knowledge of M^a_{AB} in an equilibrium state. This matrix involves up to second derivatives in dissipative variables ζ^{ab}, then we will consider generating functions only up to terms of second order in the ζ^{ab}, higher order terms will not contribute to the principal parts of the equations at equilibrium. The most general expression up to second order in ζ^{ab} for χ is

$$\chi = \chi_0(\zeta, \mu) + \chi_1(\zeta, \mu)\zeta^{ab}u_a u_b + \sum_{i=1}^{3} \chi_2^{(i)}(\zeta, \mu) S^{(i)}_{abcd}\zeta^{ab}\zeta^{cd} \tag{3.3}$$

with

$$\begin{aligned}
u^a &= \frac{\zeta^a}{\mu} \\
q^{ab} &= g^{ab} + u^a u^b \\
S^{(1)}_{abcd} &= q_{a(c}q_{d)b} - \tfrac{1}{3} q_{ab} q_{cd} \\
S^{(2)}_{abcd} &= u_{(a}q_{b)(c}u_{d)} \\
S^{(3)}_{abcd} &= \frac{3}{4}\left(\frac{q_{ab}}{3} + u_a u_b\right)\frac{3}{4}\left(\frac{q_{cd}}{3} + u_c u_d\right).
\end{aligned}$$

This follows since the right-hand side of (3.3) is the most general scalar function that can be constructed as a local function of ζ, ζ^a, g^{ab} and up to second order in ζ^{ab}; and the $S^{(i)}_{abcd}$, $(i = 1, 2, 3)$ produce the most general split of a symmetric trace-free tensor ζ^{ab} around a timelike direction u^a.

As stated before, to determine a unique fluid theory of divergence type we have to specify a generating function χ (that is, in this case five functions χ_0, χ_1 and $\chi_2^{(i)}$ $(i = 1, 2, 3)$) and a dissipation-source tensor I_A. So to describe a dilute gas with a divergence type theory we have to give the appropriate functions χ and I_A. However, since we are only interested in knowing whether this theory is causal or not and for that we need only to know the generating function χ, the dissipation-source tensor I_A is irrelevant and will not be considered from now on.

The restriction that the right-hand side of (3.1) be totally symmetric imposes conditions on the generating function χ, that is conditions on functions χ_0, χ_1, $\chi_2^{(i)}$, $(i = 1, 2, 3)$. At equilibrium, the functions χ_0 and χ_1 have to satisfy

$$m^2 \chi_{0,\zeta\mu} = \left(\chi_{1,\mu} + \frac{4}{\mu}\chi_1\right). \tag{3.4}$$

This expression gives us χ_1 if χ_0 is known. If we impose the condition that the right-hand side of (3.1) be symmetric not only at equilibrium, but also up to first order in dissipative variables, we obtain the following conditions† for $\chi_2^{(i)}$:

$$\chi^{(1)}_{2,\mu} = -\frac{1}{\mu}\left(\chi_2^{(1)} + \frac{\chi_2^{(2)}}{2}\right) \tag{3.5}$$

$$\chi^{(2)}_{2,\mu} = -\frac{m^2}{2}\frac{\chi_{1,\zeta}}{\mu} - \frac{2}{\mu}\left(\chi_2^{(2)} + \frac{3}{2}\chi_2^{(3)}\right) \tag{3.6}$$

$$\chi^{(3)}_{2,\mu} = \frac{m^2}{2}\chi_{1,\zeta\mu} - \frac{4}{\mu}\left(\frac{2}{3}\chi_2^{(2)} + \chi_2^{(3)}\right) \tag{3.7}$$

$$\chi_2^{(3)} - \chi_2^{(2)} = \frac{10}{3}\chi_2^{(1)} + \frac{m^2}{2}\chi_{1,\zeta} \tag{3.8}$$

† Equations (3.4)–(3.8) are related to equations (A4) of [5].

where it can be seen that equation (3.8) is an integrability equation for (3.5)–(3.7). Knowledge of χ_0 is thus sufficient to determine χ_1 and $\chi_2^{(i)}$ $(i = 1, 2, 3)$.

From these equations it can be seen that the main idea in [5] is the following: knowledge of an equilibrium fluid state (via kinetic theory) could give information about nearby non-equilibrium states if we impose the condition that the whole theory be of divergence type. This is because the function χ_0 can be determined by kinetic theory and then χ_1 is calculated from (3.4), and $\chi_2^{(i)}$, $(i = 1, 2, 3)$ from (3.5)–(3.7). With this generating function it is possible to build the matrix M^a_{AB} at equilibrium. The main result of our work is to prove that for a dilute gas, this system of equations is not only hyperbolic, but also causal, in the sense given above.

4. Causality of dilute gases

The function χ_0 only contains information about equilibrium fluid states. This information is sufficient to construct the complete generating function χ up to second order in dissipative variables, integrating system (3.4)–(3.7). In this section we integrate this system with a function χ_0 obtained from kinetic theory, as we show below. Finally we study causal properties of the theory obtained.

It is known [4] that at equilibrium we have

$$\mu = \frac{1}{T} \qquad u^a = \frac{N^a}{\sqrt{-N^b N_b}} \qquad \zeta = \alpha.$$

In these states, all the moments, $J^{a_1\cdots a_n}|_{\rm E}$ (where $|_{\rm E}$ means evaluation in an equilibrium state), can be calculated from the distribution function at this equilibrium state. For instance, for the first moment we have [5]

$$N^a|_{\rm E} = J^a|_{\rm E} = nu^a \qquad \text{with} \qquad n = 4\pi\eta m^3 \int_0^\infty \frac{\sinh^2(r)\cosh(r)}{\exp(-z + \gamma\cosh(r)) - \epsilon}\, {\rm d}r$$

where $z = \alpha/k$, $\gamma = m/(kT)$.

However, from divergence type theories restrictions, $N^a|_{\rm E}$ is related to the function χ_0 as follows

$$N^a|_{\rm E} = nu^a = -\chi_{0,\zeta\mu} u^a.$$

So from kinetic theory, the function $\chi_{0,\zeta\mu}$ in terms of the equilibrium distribution function is given by

$$\chi_{0,\zeta\mu} = -4\pi\eta m^3 \int_0^\infty \frac{\sinh^2(r)\cosh(r)}{\exp(-z + \gamma\cosh(r)) - \epsilon}\, {\rm d}r.$$

With this input we can solve the system (3.4)–(3.8) or equivalently the following system written in terms of dimensionless variables z and γ:

$$(\gamma^4 \chi_1)_{,\gamma} = \frac{m^2}{k}\gamma^4 \chi_{0,z\gamma}$$

$$\tilde{\chi}_{2,\gamma} = -2\frac{m^2}{k}\frac{\chi_{1,z}}{\gamma}$$

$$(\gamma^6 \chi_2^{(2)})_{,\gamma} = \gamma^5 (\gamma \tilde{\chi}_2)_{,\gamma}$$

$$\chi_2^{(3)} = \frac{4}{3}\chi_2^{(2)} - \frac{1}{3}\tilde{\chi}_2 + \frac{m^2}{2k}\chi_{1,z}$$

with $\tilde{\chi}_2 \equiv \chi_2^{(2)} - 10\chi_2^{(1)}$. The solution can be written as

$$\chi_{0,z\gamma} = -4\pi\eta m^2 k^2 J_{2,1} \tag{4.1}$$

$$\chi_1 = 4\pi\eta m^4 k\left[\frac{1}{3}\gamma J_{4,1} + \frac{1}{\gamma^4}A_1(z)\right] \tag{4.2}$$

$$\chi_2^{(1)} = 4\pi\eta m^6\left[\frac{1}{15}J_{6,0} - \frac{1}{8\gamma^4}A_{1,z}(z) - \frac{1}{6}A_2(z) + \frac{1}{5\gamma^6}A_3(z)\right] \tag{4.3}$$

$$\chi_2^{(2)} = 4\pi\eta m^6\left[\frac{2}{3}(J_{6,0} + J_{4,0}) - \frac{3}{4\gamma^4}A_{1,z}(z) + \frac{1}{3}A_2(z) + \frac{2}{\gamma^6}A_3(z)\right] \tag{4.4}$$

$$\chi_2^{(3)} = 4\pi\eta m^6\left[\left(\frac{8}{9}J_{6,0} + \frac{4}{3}J_{4,0} + \frac{1}{2}J_{2,0}\right) - \frac{2}{3\gamma^4}A_{1,z}(z) - \frac{2}{9}A_2(z) + \frac{8}{3\gamma^6}A_3(z)\right] \tag{4.5}$$

where

$$J_{n_1,n_2} = \int_0^\infty \frac{\sinh^{n_1}(r)\cosh^{n_2}(r)}{\exp(-z+\gamma\cosh(r)) - \epsilon}\,\mathrm{d}r \tag{4.6}$$

with n_1, n_2 non-negative integers, and $A_i(z)$ $(i = 1, 2, 3)$ integration constants in variable γ.

Once we have the generating function χ, we can compute the corresponding matrix $M^a_{AB}|_E$, and study its causal properties. As stated above, causality means that the characteristic velocities are less than the speed of light. So the rest of our work will be concerned with calculating these characteristic velocities for the system associated with the matrix M^a_{AB}. If there exists a vector ω^a such that $\det(\omega_a M^a_{AB}) = 0$ then this vector is normal to a characteristic surface. The condition of causality means that this surface is timelike, that is ω^a is spacelike. If we write $\omega^a = vu^a + v^a$ with $v^a v_a = 1$ and $u^a v_a = 0$, then the system is causal if the characteristic velocities v satisfies $v^2 < 1$. We will prove here this last statement for the solution (4.1)–(4.5).

The matrix $N_{AB} \equiv -\omega_a M^a_{AB}|_E$ takes a particularly simple form when one chooses the following basis:

$$\delta\varphi = \{\delta z, \delta\gamma, \delta\zeta_\perp, \delta u^a, \delta\zeta^a_\perp, \delta\zeta^{ab}_\perp\}$$

where the symbol δ indicate small variation from an equilibrium state, and

$$\delta\zeta_{\perp ab} = S^{(1)}_{abcd}\delta\zeta^{cd}$$
$$\delta\zeta_{\perp a} = -2S^{(2)}_{abcd}u^b\delta\zeta^{cd}_\perp$$
$$\delta\zeta_\perp = S^{(3)}_{abcd}\left(\frac{q^{ab}}{3} + u^a u^b\right)\delta\zeta^{cd}.$$

We still have the freedom to choose the coordinates to write the system of equations. If we choose the spacelike vector ω^a as we said before

$$\omega^a = vu^a + v^a$$

with $u^a v_a = 0$, $v^a v_a = 1$, then we can choose locally the coordinate x^0 as the integral lines of u^a, and x^1 as the integral lines of v^a. So we can have

$$\delta\varphi^A = \{\delta z, \delta\gamma, \delta\zeta_\perp, \delta u^1, \delta\zeta^1_\perp, \delta\zeta^{11}_\perp, \delta u^2, \delta\zeta^2_\perp, \delta\zeta^{12}_\perp, \delta u^3, \delta\zeta^3_\perp, \delta\zeta^{13}_\perp, \delta\zeta^{23}_\perp, \delta\zeta^{22}_\perp\}.$$

In this basis, the matrix N_{AB} takes the following block diagonal form:

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & 0 & 0 & 0 \\ 0 & \mathbf{N}_2 & 0 & 0 \\ 0 & 0 & \mathbf{N}_2 & 0 \\ 0 & 0 & 0 & \mathbf{N}_3 \end{bmatrix}$$

where the submatrix $\mathbf{N}_1$, is defined as

$$\mathbf{N}_1 = \begin{bmatrix} vn_{,z} & vn_{,\gamma} & -\frac{v}{mk}\chi_{1,z\gamma} & n & \frac{2}{mk}\frac{\chi_{1,z}}{\gamma} & 0 \\ vn_{,\gamma} & -\frac{v}{m}\rho_{,\gamma} & -\frac{v}{mk}\chi_{1,\gamma\gamma} & -\frac{1}{m}(\rho+p) & \frac{1}{\gamma}g_1 & 0 \\ -\frac{v}{mk}\chi_{1,z\gamma} & -\frac{v}{mk}\chi_{1,\gamma\gamma} & -\frac{v}{m^3}\chi^{(3)}_{2,\gamma} & -\frac{4}{3}g_1 & \frac{4}{\gamma}g_2 & 0 \\ n & -\frac{1}{m}(\rho+p) & -\frac{4}{3}g_1 & \frac{v}{m}\gamma(\rho+p) & -vg_1 & \frac{4}{3}\frac{1}{mk}\frac{\chi_1}{\gamma} \\ \frac{2}{mk}\frac{\chi_{1,z}}{\gamma} & \frac{1}{\gamma}g_1 & \frac{4}{\gamma}g_2 & -vg_1 & -\frac{2v}{m^3}\chi^{(2)}_{2,\gamma} & \frac{4}{3\gamma}g_3 \\ 0 & 0 & 0 & \frac{4}{3}\frac{1}{mk}\frac{\chi_1}{\gamma} & \frac{4}{3\gamma}g_3 & -\frac{4}{3}\frac{v}{m^3}\chi^{(1)}_{2,\gamma} \end{bmatrix}$$

with

$$g_1(z,\gamma) = \frac{2}{mk}\left(\chi_{1,\gamma} - \frac{\chi_1}{\gamma}\right)$$

$$g_2(z,\gamma) = \frac{1}{m^3}\left(\frac{2}{3}\chi_2^{(2)} + \chi_2^{(3)}\right)$$

$$g_3(z,\gamma) = \frac{1}{m^3}(\chi_2^{(2)} + 2\chi_2^{(1)}).$$

The matrices $\mathbf{N}_2$ and $\mathbf{N}_3$ are defined as follows.

$$\mathbf{N}_2 = \begin{bmatrix} \frac{v}{m}\gamma(\rho+p) & -vg_1 & \frac{1}{mk}\frac{\chi_1}{\gamma} \\ -vg_1 & -\frac{2v}{m^3}\chi^{(2)}_{2,\gamma} & \frac{1}{\gamma}g_3 \\ \frac{1}{mk}\frac{\chi_1}{\gamma} & \frac{1}{\gamma}g_3 & -\frac{v}{m^3}\chi^{(1)}_{2,\gamma} \end{bmatrix}$$

$$\mathbf{N}_3 = \begin{bmatrix} -\frac{v}{m^3}\chi^{(1)}_{2,\gamma} & 0 \\ 0 & -\frac{v}{m^3}\chi^{(1)}_{2,\gamma} \end{bmatrix}.$$

The characteristics velocities can be calculated splitting matrix N_{AB} by these blocks. It is obvious that the characteristic velocities corresponding to $\mathbf{N}_3$ vanish. Next consider the blocks associated with $\mathbf{N}_2$. It is a 3×3 determinant, so it leads to three possible values for v. One of these is zero. The other two have the following expression:

$$(v_\mathrm{T})^2 = \frac{2m^6kg_1g_3\chi_1 + m^6k^2\gamma(\rho+p)(g_3)^2 - 2m^2(\chi_1)^2\chi^{(2)}_{2,\gamma}}{k^2\gamma^2\chi^{(1)}_{2,\gamma}(2\gamma(\rho+p)\chi^{(2)}_{2,\gamma} + m^4(g_1)^2)}. \tag{4.7}$$

These characteristic velocities are called transverse, because they are related to the propagation of transverse perturbations, that is in our case, perturbations in the x^2 and x^3 directions.

There are six characteristics velocities associated with matrix $\mathbf{N}_1$. These characteristic velocities are called longitudinal, because they are related to the propagation of longitudinal perturbations, that is in our case, perturbations in the x^2 direction. It can be seen that two of these are zero, and the others are solution of an expression of the form

$$B_2(v_\mathrm{L})^4 + B_1(v_\mathrm{L})^2 + B_0 = 0 \tag{4.8}$$

with B_i ($i = 0, 1, 2$) complicated expressions of the generating function. So we have to prove that all characteristic velocities satisfy $v^2 < 1$. In the next two subsections we will show that this is the case for divergence type theories in which the equilibrium states correspond to Boltzmann, Fermi or Bose dilute gases.

Before presenting our principal results on the characteristic velocities of these theories, we have to comment on the choice of constants of integration (constant in γ but functions of z) presented in (4.2)–(4.5). It can be seen that different selections of functions $A_1(z)$ and $A_3(z)$ give us different physical theories. If we calculate the corresponding characteristic

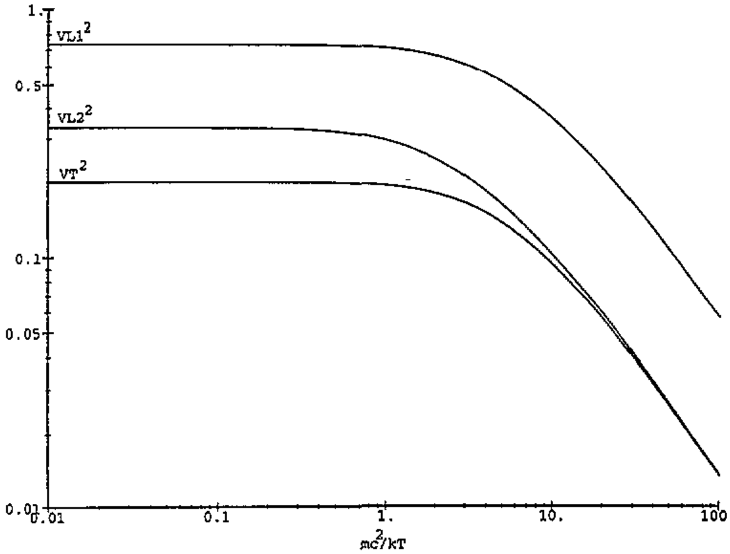

Figure 1. The longitudinal (v_{L1}, v_{L2}) and transverse (v_T) characteristic velocities, as functions of $\gamma = mc^2/kT$ for Boltzmann's dilute gas. The asymptotic formulae given in the text determine v for all γ. It is possible to draw similar plots for Fermi and Bose dilute gases.

velocities for each selection of these functions then it can be checked that the only choice which gives us the appropriate classical limit (that is $\gamma \to \infty$) is $A_1(z) = A_3(z) = 0$. The other function, $A_2(z)$, does not affect any physical property of the fluid, because it does not appear in the matrix M^a_{AB}. This is a consequence of the following: the only coefficients without γ-derivatives of $\chi_2^{(i)}$ ($i = 1, 2, 3$) are expressions involving functions g_2 or g_3. However, these functions do not depend on $A_2(z)$. So, from now on, we will assume that functions $A_i(z)$ ($i = 1, 2, 3$) are all zero.

4.1. Results for Boltzmann equilibrium distribution function

We recover the Boltzmann case by letting $\varepsilon \to 0$ in the above expressions, that is we consider the following distribution function:

$$f(x^a, p_a) = \eta \exp\left(\frac{\zeta}{k} + \frac{1}{kT} u^a p_a\right). \tag{4.9}$$

In this case the integrals $J_{m,n}$ can be expressed [5] in terms of modified Bessel function of the second kind:

$$K_n(\gamma) = \int_0^\infty \cosh(nr) \mathrm{e}^{-\gamma \cosh(r)} \, \mathrm{d}r$$

and it is possible to write a simple expression for the generating function in terms of these modified Bessel functions:

$$\chi_{0,z\gamma} = -4\pi \eta m^2 k^2 \frac{K_2}{\gamma} \mathrm{e}^z \tag{4.10}$$

$$\chi_1 = 4\pi \eta m^4 k \frac{K_3}{\gamma} \mathrm{e}^z \tag{4.11}$$

$$\chi_2^{(1)} = 4\pi\eta m^6 \frac{K_3}{\gamma^3} e^z \tag{4.12}$$

$$\chi_2^{(2)} = 4\pi\eta m^6 2\left(\frac{K_4}{\gamma^2} - \frac{K_3}{\gamma^3}\right) e^z \tag{4.13}$$

$$\chi_2^{(3)} = 4\pi\eta m^6 \left(2\frac{K_4}{\gamma^2} + \frac{4}{3}\frac{K_3}{\gamma^3} + \frac{1}{2}\frac{K_3}{\gamma}\right) e^z. \tag{4.14}$$

With these expressions for the generating function it is straightforward to compute the characteristic velocities. For example, expression (4.7) for transversal velocities can be written as

$$(v_T)^2 = \frac{G}{\gamma + 6G}\left[1 - \frac{\gamma^2 G^2}{\gamma^2 + 5G\gamma - 6G^2}\right]$$

with $G \equiv K_3/K_2$. In the same way we can solve (4.8) and obtain two roots for v_L^2 as a function of modified Bessel functions of the second kind. This expression is too complicated to give here, but its behaviour as function of $\gamma = mc^2/kT$ is shown in figure 1. It is also shown there the behaviour of v_T^2. All these results are in correspondence with previous ones in Stewart [9] and Seccia and Strumia [8]. For example in the ultra-relativistic limit ($\gamma \to 0$), we obtain

$$v_T^2 \simeq \tfrac{1}{5}c^2 \qquad v_{L1}^2 \simeq \tfrac{3}{5}c^2 \qquad v_{L2}^2 \simeq \tfrac{1}{3}c^2$$

where we prefer to write explicitly the light velocity c to clarify units. Recall that throughout this work we use the units $c = 1$. Another special case of interest is the classical limit ($\gamma \to \infty$) which gives

$$v_T^2 \simeq \frac{7}{5}\frac{k}{m}T \qquad v_{L1}^2 \simeq 5.18\frac{k}{m}T \qquad v_{L2}^2 \simeq 1.35\frac{k}{m}T.$$

All these results agree with previous ones.

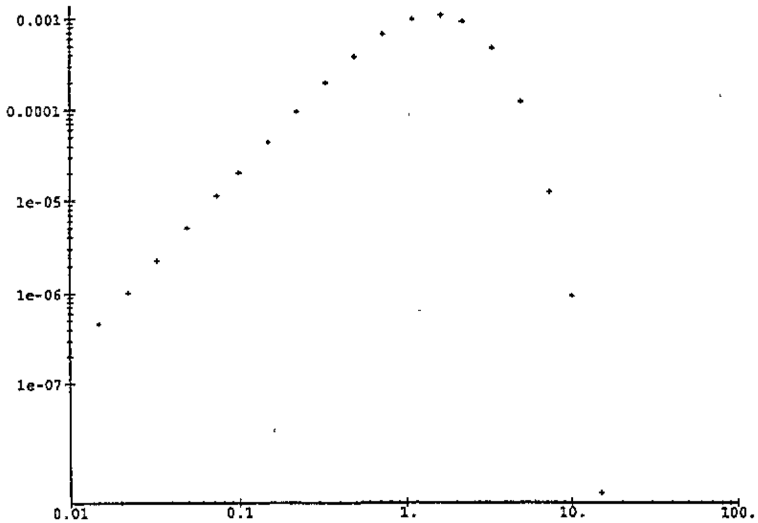

Figure 2. The difference $(v_{L1})^2|_{\rm Fermi} - (v_{L1})^2|_{\rm Bose}$ as a function of $\gamma = mc^2/kT$. It can be seen that the maximum difference is reached when $\gamma \sim 1$ that is when $mc^2 \sim kT$.

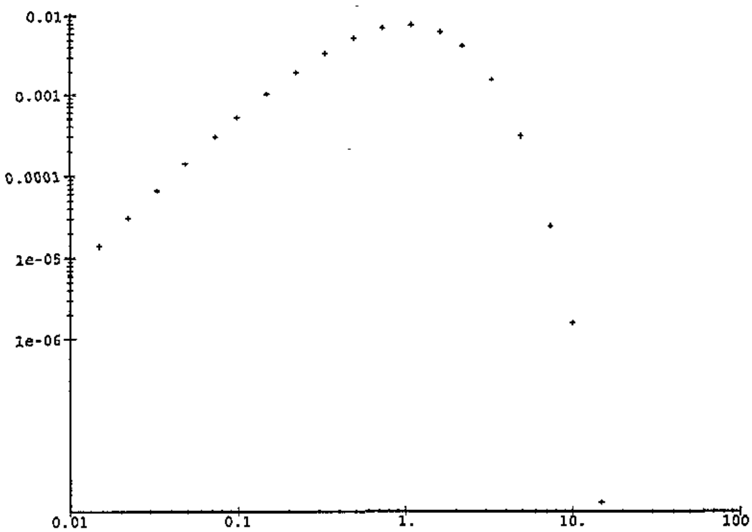

Figure 3. The difference $(v_{L2})^2|_{\rm Fermi} - (v_{L2})^2|_{\rm Bose}$ as a function of $\gamma = mc^2/kT$.

4.2. *Results for the Bose and Fermi equilibrium distribution functions*

We now consider the equilibrium distribution function as given by (2.11). In this case it is not even known whether the resultant divergence type theory is hyperbolic. In this subsection we prove causality of these divergence type theory, in the following way. First, we insert in (4.7) and (4.8) the expression for the generating function given in (4.1)–(4.5), where $\varepsilon = 1$ or -1 gives us a Bose or Fermi equilibrium distribution function, respectively. Second, we evaluate numerically the resultant expression for the characteristic velocities and confirm that they are strictly less than c. In these cases, that is Fermi and Bose statistics, we plot the characteristics velocities as a function of γ, obtaining a very similar graph to that shown in figure 1 for Boltzmann statistics. Since they are very similar to the values for Boltzmann's we just plot their differences. For example we show in figure 2 the difference $(v_{L1})^2|_{\rm Fermi} - (v_{L1})^2|_{\rm Bose}$. In figure 3 we do the same for v_{L2}. We can see that the maximum difference is reached when $\gamma \sim 1$ that is when $mc^2 \sim kT$. We also plot in figures 4 and 5 the difference $(v_{L1})^2|_{\rm Fermi} - (v_{L1})^2|_{\rm Boltzmann}$ and $(v_{L1})^2|_{\rm Boltzmann} - (v_{L1})^2|_{\rm Bose}$, respectively. All the plots were done for the case $z = 0$, for if $|z| \gg 0$, the plots become indistinguishable from those obtained with Boltzmann's statistics (see the appendix).

The limiting values of the characteristic velocities when $\gamma \to \infty$ or $\gamma \to 0$ for both statistics are the same as those we obtained for Boltzmann statistics. This can be seen analytically as follows. The fact that the classical limit of characteristic velocities for Fermi and Bose statistics gives the same results as those provided by Boltzmann statistics can be seen by the argument in the first point of the appendix. There we give an alternative form for the generating function (4.1)–(4.5). From the expressions given in the appendix it is easy to check that in the limit $\gamma \to \infty$ we reobtain the values for the characteristic velocities calculated for a Boltzmann dilute gas.

The second one, that is the ultra-relativistic limit, is explained in the second point of the appendix. The principal idea was to rewrite the functions $J_{m,n}$ which appear in the generating function separating their divergent parts when $\gamma \to 0$. We calculate the characteristic velocities with these expressions, all divergences cancelled and we finally obtain the same values as for Boltzmann's dilute gas.

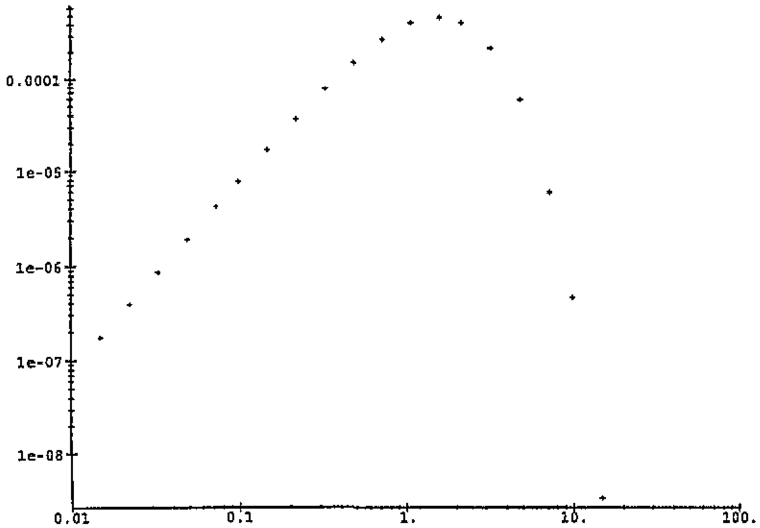

Figure 4. The difference $(v_{L1})^2|_{\text{Fermi}} - (v_{L1})^2|_{\text{Boltzmann}}$ as a function of $\gamma = mc^2/kT$. Note that the characteristic velocity for Fermi's gas is greater than for Boltzmann's gas.

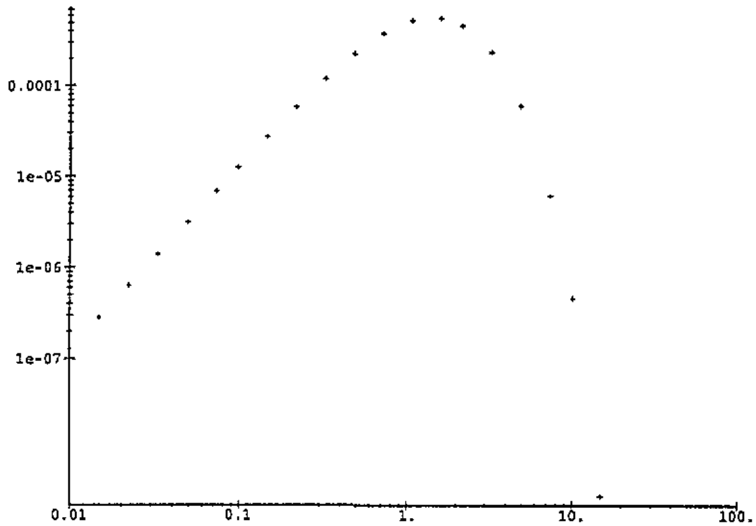

Figure 5. The difference $(v_{L1})^2|_{\text{Boltzmann}} - (v_{L1})^2|_{\text{Bose}}$ as a function of $\gamma = mc^2/kT$. Note that the characteristic velocity for Boltzmann's gas is greater than for Bose's gas.

5. Conclusions

From both an aesthetic and economic, that is physical, point of view, one would like to find the simplest theory to describe a given physical system. Candidates with this property for describing dissipative relativistic fluids are the divergence type theories. As we reviewed in section 2, a dissipative relativistic fluid divergence-type theory is completely determined by a generating function χ and a dissipation-source tensor I_{ab}. The function χ generates the principal part of the dynamical equations, giving an expression which is particularly simple and relatively easy to analyse. However, are these simple theories general enough to describe those fluids correctly in some relevant regime? When they do describe them, do they have the hyperbolicity and causality properties that on physical grounds would be expected? Here we have tested these theories with the simplest system in this subject, that is a dilute gas. The first of these questions has been answered, but from a slightly different point of view in [5]. In our treatment we derive an equation for χ, equation (3.2), which expresses the identification, outside the equilibrium of a three-tensor constructed from χ and the symmetric third momentum tensor of kinetic theory. We solved this equation up to second order in the dissipative variables—that is all that is needed to show causality near equilibrium—and note that these terms are completely determined† by the values of the generating function at equilibrium. That is the above identification selects a class of solutions whose causal behaviour seems to be determined completely from the value of measurable equilibrium quantities. We believed this to be an interesting conjecture even for regions away from equilibrium: The above identification, equation (3.2), determines a class of theories with the property that their principal part equations are completely determined from the values of χ at equilibrium. We have already resolved this conjecture in some interesting cases and the results will be published elsewhere.

The second question raised above is the main result of this work; we could show that the divergence theory which represents Bose and Fermi dilute gases are causal in the sense given in section 2. This was done by a straightforward calculation of the characteristics velocities of system given by matrix $\mathbf{N}$, in the particular basis chosen. We found that all these velocities were positive and less than c. Due to the fact that the expression for these velocities are very complicated, we calculated the asymptotic expression when one of the variables tends to zero or infinity. For intermediate values we computed numerically these velocities. In all cases we found that they are all positive and less than c, which implies that the corresponding system of equation and so the corresponding theory, is causal.

We tested the above calculation with a known example studied by Seccia and Strumia [8], that is a Boltzmann dilute gas. Our results agree with them. These results for Boltzmann's dilute gas also agree with previous ones in Stewart [9]. This agreement is not trivial, since Stewart obtained a causal theory doing approximations directly on the distribution function. It is not trivial, *a priori*, that this theory gives the same values for the characteristic velocities.

Acknowledgments

We wish to thank O Ortiz for many discussions and helpful comments on this work, and CONICOR for financial support.

Appendix

This appendix is dedicated to detailing some calculations needed to understand results for the Bose and Fermi equilibrium distribution functions.

(i) Equation (4.1) can be written as

$$\begin{aligned}\chi_{0,z\gamma} &= -4\pi\eta m^2k^2 J_{2,1} \\ &= -4\pi\eta m^2k^2(J_{0,3}-J_{0,1}) \\ &= -4\pi\eta m^2k^2\int_0^\infty \frac{\cosh^3(r)}{\exp(-z+\gamma\cosh(r))-\varepsilon}\mathrm{d}r\end{aligned}$$

† Up to integration constants which do not affect the system.

$$
\begin{aligned}
&+4\pi\eta m^2k^2\int_0^\infty \frac{\cosh(r)}{\exp(-z+\gamma\cosh(r))-\varepsilon}\mathrm{d}r \\
&= -4\pi\eta m^2k^2\frac{1}{4}\int_0^\infty \frac{\cosh(3r)}{\exp(-z+\gamma\cosh(r))-\varepsilon}\,\mathrm{d}r \\
&\quad +4\pi\eta m^2k^2\frac{1}{4}\int_0^\infty \frac{\cosh(r)}{\exp(-z+\gamma\cosh(r))-\varepsilon}\,\mathrm{d}r \\
&= -4\pi\eta m^2k^2\frac{1}{4}\sum_{l=1}^{\infty}(-\varepsilon)^{(l-1)}\mathrm{e}^{lz}\int_0^\infty \cosh(3r)\mathrm{e}^{-\gamma\cosh(r)}\,\mathrm{d}r \\
&\quad +4\pi\eta m^2k^2\frac{1}{4}\sum_{l=1}^{\infty}(-\varepsilon)^{(l-1)}\mathrm{e}^{lz}\int_0^\infty \cosh(r)\mathrm{e}^{-\gamma\cosh(r)}\,\mathrm{d}r \\
&= -4\pi\eta m^2k^2\frac{1}{4}\sum_{l=1}^{\infty}(-\varepsilon)^{(l-1)}\mathrm{e}^{lz}(K_3(l\gamma)-K_1(l\gamma)) \\
&= -4\pi\eta m^2k^2\frac{1}{4}\sum_{l=1}^{\infty}(-\varepsilon)^{(l-1)}\mathrm{e}^{lz}\frac{K_2(l\gamma)}{l\gamma}
\end{aligned}
$$

where, in the first step, we write functions $J_{m,n}$ in terms of functions of the form $J_{0,n}$; the next step is to use trigonometric identities to write terms with $\cosh^n(r)$ in terms of $\cosh(nr)$; then we compute the asymptotic expression for large $|z|$ and finally we write this expression in terms of the modified Bessel functions of the second kind. By the same way we obtain the following expressions:

$$
\begin{aligned}
\chi_{0,z\gamma} &= -4\pi\eta m^2k^2\sum_{l=1}^{\infty}(-\varepsilon)^{(l-1)}\frac{K_2(l\gamma)}{l\gamma}\mathrm{e}^{lz} \\
\chi_1 &= 4\pi\eta m^4k\sum_{l=1}^{\infty}(-\varepsilon)^{(l-1)}\frac{K_3(l\gamma)}{l\gamma}\mathrm{e}^{lz} \\
\chi_2^{(1)} &= 4\pi\eta m^6\sum_{l=1}^{\infty}(-\varepsilon)^{(l-1)}\frac{K_3(l\gamma)}{(l\gamma)^3}\mathrm{e}^{lz} \\
\chi_2^{(2)} &= 4\pi\eta m^6 2\sum_{l=1}^{\infty}(-\varepsilon)^{(l-1)}\left(\frac{K_4(l\gamma)}{(l\gamma)^2}-\frac{K_3(l\gamma)}{(l\gamma)^3}\right)\mathrm{e}^{lz} \\
\chi_2^{(3)} &= 4\pi\eta m^6\sum_{l=1}^{\infty}(-\varepsilon)^{(l-1)}\left(2\frac{K_4(l\gamma)}{(l\gamma)^2}+\frac{4}{3}\frac{K_3(l\gamma)}{(l\gamma)^3}+\frac{1}{2}\frac{K_3(l\gamma)}{l\gamma}\right)\mathrm{e}^{lz}.
\end{aligned}
$$

From these expressions we can see that the first term corresponds to the generating function calculated with Boltzmann's equilibrium distribution function. These expressions together with the asymptotic behaviour of modified Bessel functions of the second kind $K_n(\gamma)$ when $\gamma \to \infty$ tells us that limiting expression for the characteristic velocities will be the same as that we calculated for Boltzmann's dilute gas.

(ii) The limiting expressions when $\gamma \to 0$, can be obtained in the following way:

$$
J_{m,n} = \frac{1}{\gamma^{(m+n)}}\int_\gamma^\infty \frac{x^n(\sqrt{x^2-\gamma^2})^{(m-1)}}{\mathrm{e}^{-z}\mathrm{e}^{x}-\varepsilon}\,\mathrm{d}x
$$

where we have only changed the variable $x = \gamma\cosh(r)$ which allows us to identify the divergent part of $J_{m,n}$ when $\gamma \to 0$. If we expand the non-divergent part of $J_{m,n}$ around

$\gamma = 0$, and retaining the leading order in γ, we have

$$J_{m,n} \simeq \frac{1}{\gamma^{(m+n)}} \int_0^\infty \frac{x^{(m+n-1)}}{e^{-z}e^x - \varepsilon}\,dx$$

and evaluating this expression in $z = 0$, we obtain a simple expression for $J_{m,n}$ in terms of the Riemann zeta function, $\zeta(n)$

$$J_{m,n}|_{z=0} \simeq \frac{1}{\gamma^{(m+n)}}\Gamma(m+n)\zeta(m+n) \qquad \varepsilon = 1$$

$$\simeq \frac{1}{\gamma^{(m+n)}}\left(1 - \frac{1}{2^{(m+n-1)}}\right)\Gamma(m+n)\zeta(m+n) \qquad \varepsilon = -1.$$

In order to calculate the characteristic velocities we also need to know

$$\left.\frac{\partial J_{m,n}}{\partial z}\right|_{z=0} \simeq \frac{1}{\gamma^{(m+n)}}\Gamma(m+n)\zeta(m+n-1) \qquad \varepsilon = 1$$

$$\simeq \frac{1}{\gamma^{(m+n)}}\left(1 - \frac{1}{2^{(m+n-2)}}\right)\Gamma(m+n)\zeta(m+n-1) \qquad \varepsilon = -1.$$

So, with these expressions we arrive at the limiting values for the characteristic velocities for Fermi and Bose statistics, giving the same results obtained for Boltzmann statistics.

References

[1] Eckart C 1940 *Phys. Rev.* **58** 919
[2] Landau L D and Lifshitz E M 1959 *Fluid Mechanics* vol 6 (London: Pergamon)
[3] Hiscock W A and Lindblom L 1985 *Phys. Rev.* D **31** 725
[4] Geroch R and Lindblom L 1990 *Phys. Rev.* D **41** 1855
[5] Liu I S, Müller I and Ruggeri T 1986 *Ann. Phys.* **169** 191
[6] Israel W and Stewart J M 1979 *Ann. Phys.* **118** 341
[7] Pinnisi S 1989 *Symposium on Kinetic Theory and Extended Thermodynamics* ed Müller and Ruggeri (Bologna: Pitagora Editrice)
[8] Seccia L and Strumia A 1990 *Cont. Mech. and Thermodynamics* **2** 151
[9] Stewart J M 1977 *Proc. R. Soc.* A **357** 59
[10] Israel W 1987 *Relativistic Fluid Dynamics* ed A Anile and Y Choquet-Bruhat (Berlin: Springer)
[11] de Groot S R, van Leeuwen W A and van Weert Ch 1980 *Relativistic Kinetic Theory* (New York: North-Holland)