LINEAR STABILITY OF A SELF-GRAVITATING COMPRESSIBLE FLUID WITH A FREE BOUNDARY

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Abstract. We consider the initial free–boundary value problem for the self-gravitating compressible three dimensional Euler equations with positive mass density at the boundary, for which we prove the linear stability of static background solutions. Our work can be summarised in two essential steps. First, we transform the free–boundary problem into a fixed-boundary problem in the usual way by using the Lagrange formulation of Euler's equations. We then write the resulting system as a first order system of evolution and constraint equations. Second, we enlarge the system including every first derivative of the fluid velocity as a new variable. This procedure leads to whole class of systems with different evolution equations. One of these systems admits a symmetric hyperbolic formulation of the evolution equations which might be useful for numerical investigations. Another of these systems allows to decouple certain evolution equations, which then can be solved independently. We prove well posedness for the linearization of these equations near a static background. This is done using known results for the initial fixed-boundary value problem for linear symmetric hyperbolic systems. The treatment of the constraints at the boundary turns out to be the most difficult part of our approach.

Key words. Fluid equations, Initial free boundary value problem, Symmetric hyperbolic equations, hyperbolic Lagrange formulation

AMS subject classifications. 76N10, 35L50, 35Q35

1. Introduction. In order to consider the initial free–boundary value problem for the self-gravitating compressible Euler equations we first introduce the independent variables, the unknown fields and the fluid equations in a bounded domain, before formulating the problem we are interested in. Let the independent variables be the Cartesian coordinates (t, x^i) in $\mathbb{R}^4$, representing the time and the position in space. Latin indices i, j, k, l, take values $1, 2, 3$, and summation over repeated indices is assumed throughout this work. Let the unknown fields be a non-negative scalar field ρ interpreted as the fluid mass density, a vector field v^i interpreted as the fluid 3-velocity, and a scalar field ϕ interpreted as the gravitational potential. Fix a state function $p(\rho)$, which relates the pressure p with the mass density ρ. Assume that the state function is twice continuously differentiable as a function of ρ. Assume, in addition, that p is a non-negative and an increasing function of ρ in the domain $[\bar\rho_0, \infty)$, where the constant $\bar\rho_0 > 0$ satisfies $p(\bar\rho_0) = 0$. Moreover consider a domain $\mathcal{D}_T = \cup_t \{t\} \times \mathcal{D}_t$, where $t \in [0, T]$ and $\mathcal{D}_t \subset \mathbb{R}^3$ is a connected, open, bounded set for each t.

The initial free–boundary problem for the self-gravitating compressible Euler equations is to find a domain $\mathcal{D}_T$ and a unique solution ρ, v^a, ϕ of system

$$D_t\rho + \rho\,\partial_i v^i = 0 \qquad \text{in} \quad \mathcal{D}_T \tag{1.1}$$

$$\rho\, D_t v_i + \partial_i p = -\rho\,\partial_i\phi, \qquad \text{in} \quad \mathcal{D}_T \tag{1.2}$$

$$\Delta\phi = 4\pi G\rho, \qquad \text{in} \quad \mathbb{R}^3 \times [0, T] \tag{1.3}$$

with $D_t := \partial_t + v^i\partial_i$ the material derivative, for the following given initial condi-

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tions

$$\rho(0,x)=\rho_0(x) \qquad v^i(0,x)=v_0^i(x), \qquad \text{on} \quad \mathcal{D}_0 \tag{1.4}$$

on a given initial domain $\mathcal{D}_0$, and satisfying on the free boundary $\mathcal{B}_T=\cup_t\{t\}\times\partial\mathcal{D}_t$, with $t\in[0,T]$, the boundary conditions

$$\rho=\bar{\rho}_0, \quad \text{on} \quad \mathcal{B}_T, \tag{1.5}$$
$$D_t f=0 \quad \text{on} \quad \mathcal{B}_T, \quad \forall f: f|_{\mathcal{B}_T}=0, \tag{1.6}$$

while the gravitational potential ϕ satisfies the boundary condition

$$\lim_{|x|\to\infty}\phi=0. \tag{1.7}$$

We have denotes by G the gravitational constant, and by $\Delta=\delta^{ij}\partial_i\partial_j$ the Laplacian, where $\delta^{ij}=\mathrm{diag}(1,1,1)$. Latin indices i, j, k, are equivalent as superscripts or subindices, in the sense that they are rised with δ^{ij} and lowered with its inverse δ_{ij}.

With the boundary condition (1.7) ϕ is given by the Newton potential

$$\phi(t,x)=-G\int_{\mathcal{D}_t}\frac{\rho(t,x')}{|x-x'|}d^3x', \tag{1.8}$$

$|\ |$ is the Euclidean norm in $\mathbb{R}^3$. In the rest of this work we use Eqs. (1.8), in order to calculate ϕ.

The boundary condition (1.6) express that the material derivative $D_t=\partial_t+v^i\partial_i$ belongs to the tangent space of $\mathcal{B}_T$ when evaluated at that border. The constant $\bar{\rho}_0$ in the boundary condition (1.5) is the particular positive constant such that the pressure vanishes, so the boundary condition (1.5) implies that the pressure at the free boundary satisfies the usual condition

$$p|_{\mathcal{B}_T}=0. \tag{1.9}$$

The name free–boundary emphasises that the domain $\mathcal{D}_T$ is part of the unknowns. The domain, and so the part of its boundary given by $\mathcal{B}_T$, are not known before computing the solution functions ρ, v^i. The pressure must vanish at this border because there is vacuum outside the fluid. The state function relates the pressure with the mass density, so the pressure vanishes at some particular value of the mass density, called $\bar{\rho}_0$. We consider states functions satisfying $\bar{\rho}_0>0$. This condition imposes a restriction on the possible state functions. An example of state function satisfying all our assumptions is $p=K\rho^\gamma-p_0$, where K,γ and p_0 are positive constants. Liquid water can be described by such state functions for appropriate values of the free constants [4]. We do not study free–boundary problems for state functions satisfying $p(0)=0$, since the mass density and possibly the fluid sound velocity vanish at the boundary, making the equations singular. Moreover the general version of equation (1.6), is $\rho\,(D_t f)=0$. Hence in the case of a vanishing boundary density, condition (1.6) might not be satisfied and it remains open whether the boundary is in fact generated by the integral curves of the velocity vector field.

The main difficulty in our problem is the presence of a free–boundary. The compressible Euler equations can be solved in the case that the boundary is fixed. By fixed we mean given in advance as a data of the problem, although not necessarily constant in time. Euler's equations can be written as a particular case of a general type of equations, called symmetric hyperbolic. The definitions of weakly, strongly, and symmetric hyperbolic systems are reviewed in the Appendix, A.1. The initial fixed-boundary value problem for quasilinear symmetric hyperbolic systems is well understood in some particular cases [8], [13], [12], [16], [17]. These cases include characteristic boundaries of constant multiplicity. This essentially means that the normal matrix, which is the matrix formed with the coefficients of the principal part of the equations involving only normal derivatives to the boundary, can have zero eigenvalues with constant multiplicity along the boundary. In that case it is known how to prescribe boundary conditions, called maximal dissipative, such that the initial fixed-boundary value problem is well posed. There are no similar results in the literature for the initial free–boundary value problem for symmetric hyperbolic systems.

Therefore, the main strategy to solve a free–boundary problem is to transform it into a new problem for different unknown functions but involving a fixed-boundary. In the case of Euler equations this can be achieved transforming the problem from Euler to Lagrange coordinates. The position in space of the fluid particles is the unknown, as usual in Lagrange formalism. A label identifying the fluid particle is the independent variable. Euler equations for the mass density and the fluid velocity translate into a second order system for the position in space of the fluid particles and the free boundary is transformed into a fixed-boundary. There is however a disadvantage with this procedure. The resulting fluid equations in the Lagrange formalism, written as a first order system, are not symmetric hyperbolic in space dimensions greater than one. We show in Sec. 2.1 that they are only weakly hyperbolic. The a priori estimates, which are at the very basis of well posedness for symmetric or strongly hyperbolic systems, cannot be constructed in weakly hyperbolic systems. There are examples of Cauchy problem for weakly hyperbolic systems with variable coefficients which are not well posed [7]. This explains a tendency in the literature; a proof of well posedness of the Cauchy or the mixed (initial fixed-boundary value) problem for a weakly hyperbolic system, when possible, involves arguments which are specific to the particular equation under study.

The linear stability of the initial free–boundary compressible Euler equations has already been shown in [9] in a slightly different context, where gravitational effects were neglected. The main idea of the proof is to consider the Euler equations in Lagrangian coordinates and to develop specific estimates for those equations. Furthermore a condition on the background solution is imposed which prevents the occurrence of the Rayleigh-Taylor instability in incompressible fluids. We come back to this result in the last section where we compare it with our main theorem.

We present a different proof for the linear stability of static solutions of a self-gravitating fluid with a free–boundary. Our guiding idea was to use as much as possible the known techniques on well posedness of the initial fixed-boundary problem for symmetric hyperbolic systems. We also translate the free–boundary problem in the Euler coordinates into a fixed-boundary problem in Lagrange coordinates. But we then enlarge the system including every first derivative of the fluid velocity as a new variable. We include as new equations the first derivative of the fluid equations in the Lagrange formalism. The integrability conditions on the new variables are also incorporated as new equations. The result is a class of enlarged systems, an example given by (2.18)–(2.30), called here boundary adapted system. It consists of evolution equations and constraint equations coming from the integrability conditions. The name constraint means that there is no time derivative in these equations.

Einstein's equations for the gravitational field, and Maxwell's equations for the electromagnetic field are examples of systems consisting of hyperbolic evolution equations and constraint equations. The mixed problem for systems of this type can be solved as follows. First, find a solution of the mixed problem for the evolution equations. Then, show that there exists a subset of all possible boundary conditions with the following property: if the constraint equations are satisfied initially, then they are satisfied for every time during the evolution. This property is called preservation (or propagation) of the constraints. We carry on this idea on Eqs. (2.18)–(2.30) to show the linear stability of static solutions. The evolution equations are only weakly hyperbolic. However, when linearized near a static solution they decouple into a symmetric hyperbolic block that can be solved separately from the rest of the system. The solutions of this subset of the unknowns are then source functions to solve the evolution equations for the rest of the variables. These equations are then ordinary differential equations. Finally we show that there exists boundary data for the evolution equations with the following properties: it is maximal dissipative for the evolution equations; it implies the preservation of the constraints, and the resulting fluid boundary is a free–boundary for the linear perturbation. This result is summarized in Theorem 3.2.

We have mentioned that the evolution equations of the boundary adapted system are still weakly hyperbolic. This might be the reason why our argument to show well posedness to perturbation of static solutions cannot be generalized to perturbations on an arbitrary background solution. In the Appendix B we show that there exists further modifications to the *evolution equations which are symmetric hyperbolic* in the Lagrange coordinates. Such a formulation has interest in its own and might be useful for numerical simulations.

The idea to find this system is the following. Given any system of evolution and constraint equations, the former are not uniquely determined. Adding any constraint equation to the evolution equations produces new evolution equations. Using this attempt one can find a system of evolution equations for the compressible fluid equations in Lagrange coordinates which is symmetric hyperbolic, Eqs. (B.1)–(B.6) in Appendix B. This evolution equations seem, at the moment, not to be well adapted to the boundary, though. That is, it is no clear how to prescribe boundary data satisfying the three properties mentioned above:

1. that it being maximal dissipative for the evolution equations;
2. it implies the preservation of the constraints,
3. and the resulting fluid boundary is a free–boundary.

We present this system here, because further modifications of the evolution equations could produce a boundary adapted system for the full nonlinear fluid equations.

The main idea needed to construct Eqs. (B.1)–(B.6) was originated in [5], where a symmetric hyperbolic Lagrange formulation for Einstein-Euler's equations was presented. However, it was not clear at that moment whether particular geometrical features of Einstein's equations made possible this system, or the same idea could be carried out in Euler's equations in the context of Newtonian theory. We show that the latter possibility is actually true. A different starting point to obtain the same system (B.1)–(B.6) is given in [6]. This reference reformulated and generalized the ideas given in [5]. It provides a general procedure to construct relativistic symmetric hyperbolic Lagrange formulations for any symmetric hyperbolic evolution equation having a nonvanishing four-vector field as variable.

This work is organized as follows. In Sec. 2.1 we write the Lagrange formulation of compressible Euler's equations has a first order system. We show that the evolution equations are weakly hyperbolic. In Sec. 2.2 we introduce the boundary adapted system, Eqs. (2.18)–(2.30). In Sec. 3 we translate the free–boundary problem into a fixed-boundary problem. We linearize it near a static solution. We prove the main result, Theorem 3.2, which states that this linearization is well posed. We discuss the main results in Sec. 4. We have added two appendices. The first one provides the background material on hyperbolic equations needed in this work. The main definitions on hyperbolic system are reviewed in Sec. A.1. The main theorem on well posedness for the initial fixed-boundary value problem for linear symmetric hyperbolic equations is reviewed in Sec. A.2. Finally, we show in Appendix B that one can construct a symmetric hyperbolic Lagrange formulation of Euler's equations.

2. Lagrange formulation. It is known that the Lagrange formulation of Euler equations in one space dimension is symmetric hyperbolic, but fail to have this property in space dimensions greater than one.

2.1. Standard Lagrange formulation. Consider the Euler equations (1.1), (1.2) in the bounded domain $\mathcal{D}_T$, neglecting any gravitational effect. Assume that the state function satisfies every condition given in Sec. 1. Denote $\nu^2 := \frac{\partial p}{\partial \rho}$, which then satisfies $\nu^2 > 0$ for $\rho \geq \rho_0$.

The compressible Euler equations (1.1), (1.2) can be written as the following symmetric hyperbolic system

$$\nu^2 D_t\rho + \nu^2\rho\,\partial_i v^i = 0, \tag{2.1}$$
$$\rho^2 D_t v^i + \nu^2\rho\,\partial^i\rho = 0, \tag{2.2}$$

by multiplying Eq. (1.1) by ν^2, and Eq. (1.2) by ρ.

The Lagrange formulation of Eqs. (2.1), (2.2) is the following. Consider the domain $D_T = [0,T]\times D$, where $T > 0$ and $D \subset \mathbb{R}^3$ is a compact set. Fix a coordinate system (t, y^a) in D_T. Latin indices a, b, c, d, take values $1, 2, 3$. The coordinates y^a label the fluid particles. The coordinate t is added for convenience in the description of a moving fluid, which is equal the time coordinate of the Euler formalism. These coordinates are the independent variables. The unknown fields are $(\hat{x}^i, {\kappa_a}^i, \hat{\rho}, \hat{v}^i)$. The functions $\hat{x}^i(t,y)$ represent the spatial position of the fluid particles with coordinates y^a at the time t, the ${\kappa_a}^i = \frac{\partial \hat{x}^i}{\partial y^a}$ are their y-derivatives, $\hat{\rho}(t,y) = \rho(t, x(t,y))$ is the fluid mass density, and $\hat{v}^i(t,y) = v^i(t, x(t,y))$ is the fluid velocity. The hats are added to emphasize that they are functions of y^a. The Jacobian of the map $\hat{x}^i(t,y)$ at time t, given by ${\kappa_a}^i$, is introduced as a variable in order that the resulting system be of first order. Denote its inverse by ${\bar{\kappa}_i}^a$, and the determinant by $\kappa = \det({\kappa_a}^i)$ and finally introduce the hatted derivative $\hat{\partial}_i = {\bar{\kappa}_i}^a\partial_a$.

The equations in Lagrangian coordinates in our presentation include evolution and constraint equations. The evolution equations are given by

$$\partial_t\hat{x}^i = \hat{v}^i, \tag{2.3}$$
$$\partial_t{\kappa_a}^i - \partial_a\hat{v}^i = 0, \tag{2.4}$$
$$\hat{\nu}^2\partial_t\hat{\rho} + \hat{\nu}^2\hat{\rho}\,\hat{\partial}_i\hat{v}^i = 0, \tag{2.5}$$
$$\hat{\rho}^2\partial_t\hat{v}_i + \hat{\nu}^2\hat{\rho}\,\hat{\partial}_i\hat{\rho} = 0, \tag{2.6}$$

while the constraint equations are given by

$$\partial_a\hat{x}^i = {\kappa_a}^i, \tag{2.7}$$
$$\partial_{[a}{\kappa_{b]}}^i = 0, \tag{2.8}$$

where $2\partial_{[a}\kappa_{b]}{}^i = \partial_a\kappa_b{}^i - \partial_b\kappa_a{}^i$. We use this notation throughout this work: an index pair between square-brackets means antisymmetrization.

This is the Lagrange formulation of Euler's equations. The Eq. (2.3) comes from the definition of the coordinates y^a, which label fluid particles. The Eq. (2.4) is the identity $\partial_{[t}\partial_{a]}\hat{x}^i = 0$. Eqs. (2.5), (2.6) are Euler equations (2.1), (2.2), written in terms of (t, y^a). The constraint equations (2.7) and (2.8) are the definition of $\kappa_a{}^i$ and its integrability condition, respectively. The fluid equations in the Euler formulation form a symmetric hyperbolic system. However the evolution equations in the associated Lagrange formulation is only weakly hyperbolic in space dimensions greater than one.

THEOREM 2.1. *The evolution equations (2.3)–(2.6) corresponding to the Lagrange formulation of the Euler equations are only weakly hyperbolic.*

Furthermore, the generalization of Eqs. (2.3)–(2.6) to any space dimensions is weakly hyperbolic for space dimensions greater than 1.

Proof. Write the evolution equations in the form $\partial_t\hat{u} = A^a\partial_a\hat{u}$, with $\hat{u}^T = (\hat{x}^i,\ \kappa_a{}^i,\ \hat{\rho},\ \hat{v}_i)$, and T meaning transpose. The matrices A^a depend of $\hat{u}$. Fix a solution $\hat{u}$, a point (t, y^a), and compute the eigenvalues and eigenvectors of the matrix $P(\hat{u},\omega) = A^a(\hat{u})\omega_a$, where $\omega_a \in \mathbb{R}^3$. Let $g_{ab} = \kappa_a{}^i\kappa_b{}^j\delta_{ij}$ be the metric induced by the transformation $\kappa_a{}^i$, and $g^{ab} = \bar{\kappa}_i{}^a\bar{\kappa}_j{}^b\delta^{ij}$ be its inverse. Assume that ω_a is unitary with respect to g^{ab}, that is $g^{ab}\omega_a\omega_b = 1$. Let $\underline{u}^T = (\underline{x}^i,\ \underline{\kappa}_a{}^i,\ \underline{\rho},\ \underline{v}_i)$ be an eigenvector of $P(\hat{u},\omega)$ with eigenvalue λ. The equation $\lambda\underline{u} = P(\hat{u},\omega)\underline{u}$ has the form

$$\lambda\underline{x}^i = 0, \tag{2.9}$$
$$\lambda\underline{\kappa}_a{}^i = \omega_a\underline{v}^i, \tag{2.10}$$
$$\lambda\underline{\rho} = -\hat{\rho}\,\omega_i\underline{v}^i, \tag{2.11}$$
$$\lambda\underline{v}_i = -\frac{\hat{\nu}^2}{\hat{\rho}}\omega_i\underline{\rho}, \tag{2.12}$$

where we denoted $\omega_i := \bar{\kappa}_i{}^a\omega_a$. The space of unknowns has dimension 16. There are three eigenvalues, $\lambda_0 = 0$ with multiplicity 12, and $\lambda_\pm = \pm\hat{\nu}$, each with multiplicity 2. The corresponding eigenvectors are

$$\underline{u}_0 = \begin{bmatrix} \underline{x}^i \\ \underline{\kappa}_a{}^i \\ 0 \\ 0 \end{bmatrix}, \quad \underline{u}_\pm = \begin{bmatrix} 0 \\ \pm\omega_a\omega^i \\ \mp\hat{\rho} \\ \hat{\nu}\omega^i \end{bmatrix}, \tag{2.13}$$

with $\omega^i := \delta^{ij}\omega_j$. There are 12 linearly independent eigenvectors $\underline{u}_0$, parametrized by the components of $\underline{x}^i$ and $\underline{\kappa}_a{}^i$. However, there are only 2 linearly independent eigenvectors $\underline{u}_\pm$, so they do not span their four dimensional eigenspace. The component of the velocity part, $\underline{v}^i$, of the eigenvectors $\underline{u}_\pm$ has only components in the direction along ω^i. Therefore, Eqs. (2.3)–(2.6) are weakly hyperbolic.

This proof can be generalized to any number of space dimensions. In this case $\omega_a \in \mathbb{R}^n$, with $n \geq 1$. Indices i, j, and a b take values $1, \cdots, n$. The eigenvectors of $P(\hat{u},\omega)$ are $\underline{u}_0$ and $\underline{u}_\pm$ given above. The eigenvector $\underline{u}_\pm$ span its eigenspace only in the case $n = 1$. That is why the system is strongly hyperbolic in space dimension 1, but it is not in higher space dimensions. In the later case it is only weakly hyperbolic.□

REMARK 1 (On the uniqness of presentation). *The evolution equations (2.3)–(2.6) are not uniquely defined, because one can add the constraint equations (2.7), (2.8) into the evolution equations and the obtain a new set of evolution equations.*

Consider the following evolution equations

$$\partial_t \hat{x}^i = \hat{v}^i, \tag{2.14}$$
$$\partial_t \kappa_a{}^i - \partial_a \hat{v}^i = \alpha_1(\partial_a \hat{x}^i - \kappa_a{}^i), \tag{2.15}$$
$$\hat{\nu}^2 \partial_t \hat{\rho} + \hat{\nu}^2 \hat{\rho}\, \hat{\partial}_i \hat{v}^i = \alpha_2 \bar{\kappa}_i{}^a (\partial_a \hat{x}^i - \kappa_a{}^i), \tag{2.16}$$
$$\hat{\rho}^2 \partial_t \hat{v}_i + \hat{\nu}^2 \hat{\rho}\, \hat{\partial}_i \hat{\rho} = -2\alpha_3 \hat{\nu} \bar{\kappa}_i{}^a \bar{\kappa}_j{}^b \partial_{[a} \kappa_{b]}{}^j. \tag{2.17}$$

However, this freedom in the definition of the evolution equations cannot be used to find strongly hyperbolic evolution equations.

Lemma 2.2. *The evolution equations (2.14)–(2.17) are weakly hyperbolic for every value of the parameters α_i, with $i = 1, 2, 3$.*

The proof is similar to the proof of Theorem 2.1, and is not reproduced here.

2.2. Boundary adapted system. We present now, the new formulation of the Euler equation, which we announced in the introduction. First consider the domain $D_T = [0, T] \times D$, where $T > 0$ and $D \subset \mathbb{R}^3$ is a connected, open, bounded set. Let (t, y^a) be coordinates in D_T, with y^a labeling the fluid particles.

The boundary adapted system is then the following. The unknown fields are $(\hat{x}^i,\ \kappa_a{}^i,\ \hat{\phi},\ \hat{\rho},\ \hat{v}^i,\ \hat{a}^i,\ \hat{w}_a{}^i)$. The functions $\hat{x}^i(t, y)$ are the spatial position of the fluid particles with coordinates y^a at the time t, while $\kappa_a{}^i = \frac{\partial \hat{x}^i}{\partial y^a}$ represent their y-derivatives. The function $\hat{\phi}(t, y) = \phi(t, \hat{x}(t, y))$ is the gravitational potential as a function of (t, y^a). The functions $\hat{\rho}(t, y) = \rho(t, \hat{x}(t, y))$ and $\hat{v}^i(t, y) = v^i(t, \hat{x}(t, y))$ represent the mass density and the 3-velocity, respectively. The unknown $\hat{a}^i = \partial_t \hat{v}^i$ is the material acceleration, and $\hat{w}_a{}^i = \partial_a \hat{v}^i$ the y-derivative of the fluid velocity. We use the notation $\hat{w} := \bar{\kappa}_i{}^a \hat{w}_a{}^i$, and $\hat{w}_i{}^j = \bar{\kappa}_i{}^a \hat{w}_a{}^j$, where $\bar{\kappa}_i{}^a$ is the inverse of $\kappa_a{}^i$. Fix a state function $p(\rho)$ as described in Sec. 1, and denote $\hat{p} = p(\hat{\rho})$. The evolution equations are given by

$$\partial_t \hat{x}^i = \hat{v}^i, \tag{2.18}$$
$$\partial_t \kappa_a{}^i = \hat{w}_a{}^i, \tag{2.19}$$
$$\partial_t \hat{\rho} = -\hat{\rho}\, \hat{w}, \tag{2.20}$$
$$\partial_t \hat{v}^i = \hat{a}^i, \tag{2.21}$$
$$\partial_t \hat{a}_i - \hat{\nu}^2 \hat{\partial}_i \hat{w} = -\hat{\alpha} \hat{w} (\hat{a}_i + \hat{\partial}_i \hat{\phi}) - \hat{w}_{ij} \hat{a}^j - \hat{\partial}_i (\partial_t \hat{\phi}), \tag{2.22}$$
$$\partial_t \hat{w}_{ij} - \hat{\partial}_i \hat{a}_j = -\hat{w}_i{}^k \hat{w}_{kj}, \tag{2.23}$$

where $\hat{\partial}_i = \bar{\kappa}_i{}^a \partial_a$, and the function $\hat{\alpha} = \frac{\hat{\rho}\hat{\beta}}{\hat{\nu}^2}$, with $\beta = \frac{\partial^2 p}{\partial \rho^2}$, and $\hat{\beta} = \beta(\hat{\rho})$. The gravitational potential is given by

$$\hat{\phi} = -G \int_D \frac{\hat{\rho}(t, y')}{|\hat{x}(t, y) - \hat{x}(t, y')|} g d^3 y', \tag{2.24}$$

with $g = [\det(g_{ab})]^{(\frac{1}{2})}$, where $g_{ab} = \kappa_a{}^i \kappa_b{}^j \delta_{ij}$ is the Euclidean metric δ_{ij} written in the Lagrange coordinates. The constraint equations are,

$$\partial_a \hat{x}^i = \kappa_a{}^i, \tag{2.25}$$
$$\partial_a \hat{v}^j = \hat{w}_a{}^j, \tag{2.26}$$
$$\frac{\hat{\partial}_i \hat{p}}{\hat{\rho}} + \hat{\partial}_i \hat{\phi} = -\hat{a}_i, \tag{2.27}$$
$$\partial_{[a} \kappa_{b]}{}^i = 0, \tag{2.28}$$
$$\partial_{[a} \hat{w}_{b]}{}^k = 0, \tag{2.29}$$
$$\hat{\partial}_{[i} \hat{a}_{j]} = 0. \tag{2.30}$$

These equations (2.18)–(2.30) form the boundary adapted system. Let us just outline how it can can be obtained. Compute the standard Lagrange formulation of the self gravitating Euler equations (1.1), (1.2). Then, Eq. (2.18) is the definition of y^a, that is, a label for the fluid particles. The constraint Eq. (2.25) is the definition of $\kappa_a{}^i$, and Eq. (2.28) its integrability condition. The constraint Eq. (2.26) is the definition of the new unknown $\hat{w}_a{}^i$, and Eq. (2.29) is its integrability condition. The evolution Eq. (2.19) is the evolution equation (2.4) replacing $\partial_a \hat{v}^i$ by $\hat{w}_a{}^i$. The evolution equation (2.21) is the definition of a_i as the material acceleration, then the constraint equation (2.27) is the original Euler equation (2.6) (with nonzero gravitational potential). The constraint equation (2.30) is its integrability condition.

The equations (2.22), (2.23) are obtained following the main idea in [5]. The first equation comes from the remaining integrability condition of the original Euler equations (2.5), (2.6) (and including the gravitational potential), that is,

$$\partial_t \left(\frac{\hat{\partial}_i \hat{p}}{\hat{\rho}} \right) - \hat{\partial}_i \left(\frac{\partial_t \hat{p}}{\hat{\rho}} \right) = -\frac{1}{\hat{\rho}} \hat{w}_i{}^j \hat{\partial}_j \hat{p}, \tag{2.31}$$

where we used the commutator $[\partial_t, \hat{\partial}_i] = -\hat{w}_i{}^j \hat{\partial}_j$. This commutator can be obtained noticing that the evolution equation (2.19) implies $(\partial_t \bar{\kappa}_i{}^a)\partial_a = -\hat{w}_i{}^j \hat{\partial}_j$. The expression on the left hand side can also be written as follows,

$$\partial_t \left(\frac{\hat{\partial}_i \hat{p}}{\hat{\rho}} \right) - \hat{\partial}_i \left(\frac{\partial_t \hat{p}}{\hat{\rho}} \right) = -\partial_t \hat{a}_i - \partial_t(\hat{\partial}_i \hat{\phi}) + \hat{\partial}_i(\hat{\nu}^2 \hat{w}) \tag{2.32}$$

where the last term on the right hand side comes from Euler's equation (2.5). Therefore, one obtains

$$\partial_t \hat{a}_i - \hat{\nu}^2 \hat{\partial}_i \hat{w} = (\hat{\partial}_i \hat{\nu}^2)\hat{w} + \hat{w}_i{}^j \frac{\hat{\partial}_j \hat{p}}{\hat{\rho}} - \partial_t(\hat{\partial}_i \hat{\phi}). \tag{2.33}$$

Note that $\hat{\partial}_i(\hat{\nu}^2) = \frac{\hat{\alpha}\hat{\partial}_i\hat{p}}{\hat{\rho}}$. Introducing the constraint $\frac{\hat{\partial}_i\hat{p}}{\hat{\rho}} + \hat{\partial}_i\hat{\phi} = -\hat{a}_i$ one obtains

$$\partial_t \hat{a}_i - \hat{\nu}^2 \hat{\partial}_i \hat{w} = -\hat{\alpha}\hat{w}(\hat{a}_i + \hat{\partial}_i \hat{\phi}) - \hat{w}_i{}^j(\hat{a}_j + \hat{\partial}_j \hat{\phi}) - \partial_t(\hat{\partial}_i \hat{\phi}). \tag{2.34}$$

This is Eq. (2.22) if one recalls the commutator $[\partial_t, \hat{\partial}_i] = -\hat{w}_i{}^j \hat{\partial}_j$. The Eq. (2.23) is just the identity $\partial_{[t}\partial_{a]}\hat{v}^j = 0$, written in terms of $\hat{w}_{ij}$ and $\hat{a}_i$. This finishes the procedure to obtain the boundary-adapted system.

3. Initial free–boundary problem. Consider the initial free–boundary value problem introduced in Sec. 1 for the self-gravitating compressible Euler equations. In this section we prove the linear stability of static background solutions of the boundary adapted Lagrange formulation of this problem for which we neglect perturbations in the gravitational potential. We first convert the original free–boundary problem given in Sec. 1 into a fixed-boundary problem for the boundary adapted Eqs. (2.18)–(2.30), with particular boundary conditions. These particular boundary conditions have the information that the fixed-boundary in the Lagrange formalism corresponds to a fluid free–boundary. Because of this reason, *we keep the expression free–boundary to name our problem in Lagrange coordinates, although one solves an initial fixed-boundary value problem.* We then compute the linearization of Eqs. (2.18)–(2.30) around a static background solution. We show well posedness of the initial fixed-boundary problem for the linear system.

The initial free–boundary problem for the boundary adapted Lagrange equations (2.18)–(2.30), is the following. *Find a vector valued function* $\hat{u} = (\hat{x}^i,\ \kappa_a{}^i,\ \hat{\rho},\ \hat{v}^i\ ,\ \hat{a}^i,\ \hat{w}_{ij})$*, solution in* D_T *of both the evolution Eqs. (2.18)–(2.23), where* $\hat{\phi}$ *is given by (2.24), and of the constraint equations (2.25)–(2.30). This solution must be uniquely specified in terms of some initial and boundary data of the form*

$$\hat{u}(0,y) = \hat{u}_0(y) \quad on \quad D \tag{3.1}$$

$$\hat{\rho} = \bar{\rho}_0, \quad with \quad \bar{\rho}_0 > 0 \quad p(\bar{\rho}_0) = 0 \quad on \quad \partial D \tag{3.2}$$

$$\hat{w} = 0 \quad on \quad B_T \tag{3.3}$$

Remark 2. *The solution to this problem in* D_T *defines a solution to the initial free–boundary problem for the self-gravitating compressible Euler equations (1.1)–(1.2), presented in Sec. 1, in the domain* $\mathcal{D}_T = \cup_t\{t\}\times\mathcal{D}_t$*. One has to choose* $D = \mathcal{D}_0$*. The information about the time evolution of* $\mathcal{D}_t$ *is encoded into the functions* $x^i(t,y)$*, for* $y^a \in \partial D$*. Note that the condition that the density is constant must only hold on the boundary of the initial hyersurface. The 3-surface* $\mathcal{B}_T$ *in space-time is a free boundary because* $w|_{\mathcal{B}_T} = 0$*, which implies that* ρ *is constant in that boundary. The initial condition* $\hat{\rho}|_{\partial D} = \bar{\rho}_0$ *implies that this constant is* $\rho = \bar{\rho}_0$*. Therefore,* $p|_{\mathcal{B}_T} = 0$*, and the boundary is a fluid free–boundary.*

3.1. The linearized problem. Consider a smooth static solution to the initial free–boundary value problem above. That is, fields $\hat{x}^i_0 = \delta_a{}^i y^a$, $\hat{v}^k_0 = 0$, $\hat{a}^i_0 = 0$, and $\hat{w}_{0ij} = 0$, where we have chosen that the x and y coordinates coincide. The fields $\hat{\rho}_0$ and $\hat{\phi}_0$ satisfies the equation

$$\frac{\hat{\partial}_i\hat{p}_0}{\hat{\rho}_0} = -\hat{\partial}_i\hat{\phi}_0 \tag{3.4}$$

and $\hat{\phi}_0$ is given by (1.8). Note that $\partial_t\hat{\phi}_0 = 0$. Denote by $\nu_0 = \nu(\hat{\rho}_0)$. Introduce the unknowns

$$\check{x}^i = \hat{x}^i - \delta_a{}^i y^a, \quad \check{\kappa}_a{}^i = \kappa_a{}^i - \delta_a{}^i, \tag{3.5}$$

$$\check{\rho} = \hat{\rho} - \hat{\rho}_0, \quad \check{v}^i = \hat{v}^i, \quad \check{a}^i = \hat{a}^i, \quad \check{w}_{ij} = \hat{w}_{ij}. \tag{3.6}$$

Neglect the gravitational effects of the perturbation, that is, set $\check{\phi} = 0$.

The linearization of the boundary adapted system (2.18)–(2.30) is the following:

$$\partial_t\check{x}^i = \check{v}^i, \tag{3.7}$$

$$\partial_t\check{\kappa}_a{}^i = \check{w}_a{}^i, \tag{3.8}$$

$$\partial_t\check{\rho} = -\hat{\rho}_0\,\check{w}, \tag{3.9}$$

$$\partial_t\check{v}^i = \check{a}^i, \tag{3.10}$$

$$\partial_t\check{a}_i - \nu_0^2\hat{\partial}_i\check{w} = -\hat{\alpha}(\hat{\partial}_i\hat{\phi}_0)\check{w}, \tag{3.11}$$

$$\nu_0^2\partial_t\check{w} - \nu_0^2\hat{\partial}_i\check{a}^i = 0, \tag{3.12}$$

$$\partial_t\check{w}_{[ij]} = 0, \tag{3.13}$$

$$\partial_t\check{w}_{\langle ij\rangle} - \hat{\partial}_{\langle i}\check{a}_{j\rangle} = 0, \tag{3.14}$$

where $\hat{\partial}_i = \delta_i{}^a\partial_a$, and $\check{w}_{\langle ij\rangle} = \check{w}_{(ij)} - \delta_{ij}\frac{\check{w}}{3}$, that is the symmetric trace-free part of $\check{w}_{ij}$. The linearized constraint equations take the form

$$\partial_a \check{x}^i = \check{\kappa}_a{}^i, \qquad (3.15)$$
$$\partial_a \check{v}^j = \check{w}_a{}^j, \qquad (3.16)$$
$$\left(\frac{\hat{\partial}_i p}{\rho}\right)\check{} = -\check{a}_i, \qquad (3.17)$$
$$\partial_{[a}\check{\kappa}_{b]}{}^i = 0, \qquad (3.18)$$
$$\partial_{[a}\check{w}_{b]}{}^k = 0, \qquad (3.19)$$
$$\hat{\partial}_{[i}\check{a}_{j]} = 0. \qquad (3.20)$$

The initial free–boundary value formulation for the linearized Lagrange equations is the following.

Definition 3.1. *Find a vector valued function $\check{u} = (\check{x}^i,\ \check{\kappa}_a{}^i,\ \check{\rho},\ \check{v}^i,\ \check{a}^i,\ \check{w}_{ij})$, solution in D_T of both the evolution Eqs. (3.7)–(3.14), and of the constraint equations (3.15)–(3.20). This solution must be uniquely specified in terms of the initial and boundary data*

$$\check{u}(0,y) = \check{u}_0(y) \qquad on \quad D \qquad (3.21)$$
$$\check{\rho} = \bar{\rho}_0, \quad with \quad \bar{\rho}_0 > 0, \quad p(\bar{\rho}_0) = 0 \qquad on \quad \partial D \qquad (3.22)$$
$$\check{w} = 0 \qquad on \quad B_T. \qquad (3.23)$$

The main result is the following.

Theorem 3.2 (Linear stability of the initial free–boundary problem). *Consider the initial free–boundary value problem for the linear evolution equations (3.7)–(3.14), on the domain $D_T = [0,T] \times D$. Let n_i denote the components of the outward unit normal form to ∂D, and $q_{ij} = \delta_{ij} - n_i n_j$ the orthogonal projector.*

1. *Fix as initial data on D the functions $\check{x}^i = 0$, $\check{\rho}$, $\check{v}^i$ in $H^s(D)$ for $s \geq 1$. The initial data for $\check{\kappa}_a{}^i$, $\check{w}_{ij}$, and $\check{a}^i$ are given by Eqs. (3.15), (3.16), and (3.17), respectively. Impose $\check{\rho} = \bar{\rho}_0$ at ∂D.*
2. *Prescribe as boundary data on B_T the value of the function $\check{w} = 0$.*
3. *Assume that the boundary data and the initial data satisfy the compatibility condition of order s.*

Then there exists a vector valued function $\check{u} \in C_T(H^s)$ solution of the initial free–boundary value problem 3.1.

The idea of the proof is, first, to find a solution of the mixed problem for the evolution equations (3.7)–(3.14). This is done in Lemma 3.3. These equations decouple into a symmetric hyperbolic block for the unknowns $\check{a}_i$, and $\check{w}$. The equations are (3.11), (3.12). This subsystem can be solved separately from the rest of the system. The main theorem to show well posedness of the mixed problem for this subsystem is given in [16], and it is reviewed in the Appendix A.2, Theorem A.5. The boundary data $\check{w} = 0$ at B_T is maximal dissipative for this subsystem. The solution fields, $\check{a}_i$, $\check{w}$ are then the source functions to solve the evolution equations for the rest of the variables, Eqs. (3.7)–(3.10), (3.13), (3.14). These equations are ordinary differential equations.

Second, show that constraint equations (3.15)–(3.20) are preserved along the evolution. This is done in Lemma 3.4. Given the solution found in Lemma 3.3, we introduce the constraint quantities (3.30)–(3.35). The evolution Eqs. (3.7)–(3.14) imply that these constraint quantities satisfy the equations (3.36)–(3.41). We check that these equations have a unique solution given by the zero solution.

Finally, Lemma 3.5 shows that the differentiability of the solution in the directions normal to the boundary is the same as the differentiability in the tangential directions to that boundary. The solution found in Lemma 3.3 belongs to the function space

$H^s_*(D)$ for $s \geq 1$. These spaces are defined in [16] and reviewed in the Appendix. For every element in that space the differentiability at the boundary in the directions normal to the boundary is only half the differentiability in directions tangential to the boundary. This is the differentiability of solutions to general symmetric hyperbolic systems with characteristic (but not totally characteristic) normal matrix. In our case, the constraint equations give the missing differentiability in the normal directions to the boundary. This agrees with previous results in mixed problems with fixed-boundary in Euler coordinates, given in [14] and [15]. In these works the solution has the same differentiability in the directions tangential and normal to the boundary.

Proof. of theorem 3.2 We divide the proof in the following three Lemmas.

LEMMA 3.3. *Assume the hypothesis in Theorem 3.2. Then, there exists a unique function $\breve{u} \in C_T(H^s_*)$, solution of the evolution equations (3.7)–(3.14). Moreover, there exist positive constants C_1 and C_2 such that*

$$\|\breve{u}(t)\|^2_{s,*} \leq C_1 e^{C_2 t} \|\breve{u}_0\|^2_{s,*} \tag{3.24}$$

for each $t \in [0,T]$, and $s \geq 1$.

LEMMA 3.4. *Assume the hypothesis in Theorem 3.2, and let $\breve{u}$ be the solution of the evolution equations (3.7)–(3.14) given in Lemma 3.3. Then, this vector valued function $\breve{u}$ satisfies the constraint equations (3.15)–(3.20).*

LEMMA 3.5. *Assume the hypothesis in Theorem 3.2, and let $\breve{u}$ be the solution of the evolution equations and constraint equations (3.7)–(3.20) given in Lemma 3.4. Then, the solution $\breve{u}$ belongs to $C_T(H^s)$.*

As indicated above these three lemmas imply theorem 3.2. □

Proof. (of Lemma 3.3) Consider the evolution equations (3.11)–(3.12) for the unknowns $\breve{a}_i$, $\breve{w}$. Once these two fields $\breve{a}_i$, $\breve{w}$ are known, one can use them as source functions in the ordinary differential equations Eqs. (3.7)–(3.10), (3.13), (3.14). So, the rest of the proof is focused only in (3.11), (3.12).

The boundary data $\breve{w} = 0$ given in assumption 2 of Theorem 3.2 is maximally dissipative for equations (3.11), (3.12). This can be seen solving the eigenvalue problem for the associated normal matrix. Let $\breve{z}^T = (\breve{a}^i, \breve{w})$, and write the volution equations for these unknowns as $\mathbf{A}^0\partial_t \breve{z} = \mathbf{A}^i \partial_i \breve{z} + \mathbf{B}\breve{z}$. Let $\mathbf{A}^n := \mathbf{A}^i n_i$ be the normal matrix of Eqs. (3.11), (3.12). Note that this normal matrix is negative the one introduced in Sec. A.2. Then, the eigenvalue equations, $\lambda \underline{z} = \mathbf{A}^n \underline{z}$, are the following,

$$\lambda \underline{a}_i = \nu_0^2 n_j \underline{w}, \tag{3.25}$$

$$\lambda \underline{w} = \underline{a}_n, \tag{3.26}$$

where $\underline{a}_n = \underline{a}_i n^i$. Also introduce $\underline{a}'_i = q_i{}^j \underline{a}_j$, so one has $\underline{a}_i = \underline{a}_n n_i + \underline{a}'_i$. The eigenvalues are $\lambda_0 = 0$ with multiplicity 2, and $\lambda_\pm = \pm\nu_0$, each with multiplicity 1. The corresponding eigenvectors are

$$\underline{z}^{(0)} = \begin{bmatrix} \underline{a}'_i \\ 0 \end{bmatrix}, \quad \underline{z}^{(\pm)} = \begin{bmatrix} \nu_0 n_i \\ \pm 1 \end{bmatrix} \tag{3.27}$$

Maximally dissipative boundary conditions have the form

$$\underline{z}^{(+)} = H \underline{z}^{(-)} + F \qquad \text{with} \quad H^2 \leq 1, \tag{3.28}$$

where $\underline{z}^{(\pm)} = (\alpha^{(\pm)}, \pm\alpha^{(\pm)})^T$, with $2\nu_0 \alpha^{(\pm)} = (\underline{a}_n \pm \nu_0 \underline{w})$. We have also introduced $|\;|$ as the operator norm. Choose $F = 0$ and H with $H = 1$ such that

$$\left(\alpha^{(+)} - \alpha^{(-)}\right) = \underline{w}, \tag{3.29}$$

therefore, the prescription of $\underline{w} = 0$ is maximally dissipative for equations (3.11), (3.12). In addition, the initial data belongs to $H^s(D)$, so it belongs to $H^s_*(D)$. Therefore, Theorem A.5 implies that there exists a solution $\check{a}_i$, $\check{w} \in C_T(H^s_*)$ to Eqs. (3.11), (3.12), which satisfies the estimate (3.24). Integrating the ordinary differential equations (3.7)–(3.10), (3.13), (3.14) with $\check{a}_i$ $\check{w}$ as source functions one obtains a solution $\check{x}^i$, $\check{\kappa}_a{}^i$, $\check{v}^i$, $\check{w}_{[ij]}$, $\check{w}_{\langle i\rangle}$ in $C_T(H^s_*)$.□

Proof. (of Lemma 3.4) Preservation of the constraints. Given a solution $\check{u}$ of Eqs. (3.7)–(3.14), introduce the constraint quantities,

$$\mathcal{Q}_a{}^i := \partial_a \check{x}^i - \check{\kappa}_a{}^i, \quad (3.30) \qquad \mathcal{C}_a{}^i := \epsilon_a{}^{bc}\partial_b \check{\kappa}_c{}^i, \quad (3.33)$$

$$\mathcal{Q}_{ij} := \hat{\partial}_i \check{v}_j - \check{w}_{ij}, \quad (3.31) \qquad \mathcal{C}_{ij} := \epsilon_i{}^{kl}\hat{\partial}_k \check{w}_{lj}, \quad (3.34)$$

$$\mathcal{Q}_i := \frac{\nu_0^2}{\rho_0}\hat{\partial}_i \check{\rho} + \check{a}_i, \quad (3.32) \qquad \mathcal{C}_i := \epsilon_i{}^{jk}\hat{\partial}_j \check{a}_k. \quad (3.35)$$

Take a time derivative of every constraint quantity. Because $\check{u}$ satisfy the evolution equations (3.7)–(3.14), the associated constraint quantities satisfy the following system,

$$\partial_t \mathcal{Q}_a{}^i = \delta_a{}^j \mathcal{Q}_j{}^i, \quad (3.36) \qquad \partial_t \mathcal{C}_a{}^i = \delta_a{}^j \mathcal{C}_j{}^i, \quad (3.39)$$

$$\partial_t \mathcal{Q}_{ij} = \frac{1}{2}\epsilon_{ij}{}^k \mathcal{C}_k, \quad (3.37) \qquad \partial_t \mathcal{C}_{ij} = \frac{1}{2}\hat{\partial}_j \mathcal{C}_i. \quad (3.40)$$

$$\partial_t \mathcal{Q}_i = 0, \quad (3.38) \qquad \partial_t \mathcal{C}_i = 0, \quad (3.41)$$

The initial data used to find $\check{u}$ implies that all the constraint quantities vanish initially. Then Eq. (3.41) implies that $\mathcal{C}_i$ vanishes in D_T. Then Eq. (3.40) implies that $\mathcal{C}_{ij}$ vanishes in D_T. Therefore, all remaining constraint quantities vanish in that domain.□

Proof. (of Lemma 3.5) The proof is based in the following result.

PROPOSITION 3.6. *Assume the hypotheses in Lemma 3.4. Extend the normal vector n^a to a neighborhood of ∂D in D, as the unique solution of the geodesic equation. Then, in that neighborhood the solution $\check{u}$ satisfies*

$$n^a \partial_a \check{u} = \mathbf{A}^0 \partial_t \check{u} + \mathbf{A}^i q_i{}^j \partial_j \check{u} + \mathbf{F}, \tag{3.42}$$

where the matrices $\mathbf{A}^0$, $\mathbf{A}^i$ depend on $\check{u}$ and $\check{u}_0$, while the vector $\mathbf{F}$ depends on $\check{u}$, $\check{u}_0$, and $\partial_i \check{u}_0$, with $\check{u}_0$ the initial data.

We assume that the Proposition 3.6 is true, and we complete the proof of Lemma 3.5. The equation (3.42) holds at the boundary ∂D and also in a neighborhood of ∂D, for all $t \in [0, T]$. This means that normal derivatives of $\check{u}$ at the boundary can be computed from $\check{u}$, from tangential and time derivatives of $\check{u}$ and from the initial data $\check{u}_0$. The term $\mathbf{F}$ in the right hand side of (3.42) contains normal derivatives of the initial data, but not of the solution.

Because the equation holds in a neighborhood of ∂D we can take derivatives in the n^a direction. Commuting derivatives and Eq. (3.42) implies that second derivatives of $\check{u}$ in the normal direction can be computed in terms of tangential and time derivatives of $\check{u}$ and on second derivatives of the initial data $\check{u}_0$.

This procedure gives an expression for the s-normal derivative of $\check{u}$ in terms of s_1-tangential and s_2-time derivatives of $\check{u}$ with $s = s_1 + s_2$, and s-derivatives of the initial data. These equations holds also at the boundary ∂D. Then, if the solution belongs to $C_T(H^s_*)$ and the initial data belongs to $H^s(D)$, then the solution $\breve{u}$ belongs to $C_T(H^s)$.□

Proof. (of Proposition 3.6) We assume that $\breve{u}$ satisfies both the evolution and the constraint equations. The proof is to compute the normal derivative of $\breve{u}$ and to verify that the resulting expression is of the form given by Eq. (3.42). Introduce first the notation $\partial_n = n^a\partial_a$, and the same holds for any solution field, that is, the subindex n in any field means contraction with n^a. For example $\breve{\kappa}_n{}^i := n^a\breve{\kappa}_a{}^i$. We also introduce the space derivatives of the normal vector, $k_a{}^b = q_a{}^c\partial_c n^b$. We use the notation $\breve{u}(t)$ or $\breve{u}$ for the value of the field at time t, and $\breve{u}(0)$ for its value at time $t = 0$. Then, $\partial_n\breve{u}$ has the form,

$$\partial_n\breve{x}^i = \breve{\kappa}_n{}^i, \tag{3.43}$$

$$\partial_n\breve{v}_i = \breve{w}_{ni}, \tag{3.44}$$

$$\partial_n\breve{\rho} = -\frac{\rho_0}{\nu_0^2}\breve{a}_n, \tag{3.45}$$

$$\partial_n(q_a{}^b\breve{\kappa}_b{}^i) = q_a{}^b\partial_b(\breve{\kappa}_n{}^i) - k_a{}^b\breve{\kappa}_b{}^i, \tag{3.46}$$

$$\begin{aligned}\partial_n(q_{ij}\breve{\kappa}_n{}^j) &= \partial_n[q_{ij}\breve{\kappa}_n{}^j(0)] + t\partial_n[2q_i{}^j n^k\breve{w}_{[kj]}(0)]\\ &+[q_i{}^j\partial_j\breve{\kappa}_n{}^n(t) - k_i{}^j\breve{\kappa}_j{}^n(t)]\\ &-[q_i{}^j\partial_j\breve{\kappa}_n{}^n(0) - k_i{}^j\breve{\kappa}_j{}^n(0)],\end{aligned} \tag{3.47}$$

$$\begin{aligned}\partial_n(\breve{\kappa}_n{}^n) &= -q_i{}^a\partial_a\breve{\kappa}_n{}^i + k_i{}^a\breve{\kappa}_a{}^i + \frac{1}{\nu_0^2}\breve{a}_n\\ &+\partial_n\left[\delta_i{}^a\breve{\kappa}_a{}^i(0) + \frac{\breve{\rho}(0)}{\rho_0}\right],\end{aligned} \tag{3.48}$$

$$\partial_n(q_i{}^j\breve{a}_j) = q_i{}^a\partial_a\breve{a}_n - k_{ij}\breve{a}^j, \tag{3.49}$$

$$\partial_n\breve{a}_n = -q^{aj}\partial_a(q_{jk}\breve{a}^k) - k_i{}^i\breve{a}_n + \partial_t\breve{w}, \tag{3.50}$$

$$\partial_n\breve{w} = \frac{1}{\nu_0^2}\partial_t\breve{a}_n + \frac{\alpha_0}{\nu_0^2}\breve{w}\partial_n\phi_0, \tag{3.51}$$

$$\partial_n\breve{w}_{[ij]} = \partial_n\breve{w}_{[ij]}(0), \tag{3.52}$$

$$\partial_n(q_i{}^j\breve{w}_{jk}) = q_i{}^a\partial_a\breve{w}_{nk} - k_i{}^j\breve{w}_{jk}, \tag{3.53}$$

$$\begin{aligned}\partial_n(\breve{w}_{nj}q_i{}^j) &= q_i{}^a\partial_a\breve{w} - q_i{}^a\partial_a(q^{kl}\breve{w}_{kl}) - k_i{}^j\breve{w}_{jn} - k_i{}^j\breve{w}_{nj}\\ &+\partial_n[2q_i{}^j n^k\breve{w}_{[jk]}(0)].\end{aligned} \tag{3.54}$$

The rest of the proof is to explain how we have obtained Eqs. (3.43)–(3.54). The equations (3.43)–(3.45) are the contraction of the constraint equations (3.15)–(3.17) with the normal vector n^i, respectively.

Eq. (3.46) is obtained in the same way, that is, contract Eq. (3.18) with the normal vector n^a. The other two equations, (3.47), (3.48) require more work. We start with Eq. (3.47). Contract evolution Eq. (3.8) with $q_{ij}n^a$ and then take a normal derivative. One obtains

$$\begin{aligned}\partial_t[\partial_n(q_{ij}\breve{\kappa}_n{}^j)] &= \partial_n(q_i{}^j\breve{w}_{nj}), &(3.55)\\ &= \partial_n\left[q_i{}^j\breve{w}_{jn} + 2q_i{}^j n^k\breve{w}_{[kj]}\right], &(3.56)\\ &= q_i{}^j\partial_j\breve{w}_{nn} - k_i{}^j\breve{w}_{jn} + \partial_n(2q_i{}^j n^k\breve{w}_{[kj]}), &(3.57)\end{aligned}$$

where in the second line we used the definition of the vorticity $2\breve{w}_{[ij]} = \breve{w}_{ij} - \breve{w}_{ji}$, and the first two terms in the third line comes from the first one in the second line, because of the constraint equation (3.19) contracted with the normal vector n^i. Now, the evolution equation (3.13) implies that $\breve{w}_{[ij]}(t) = \breve{w}_{[ij]}(0)$. Also use evolution Eq. (3.8) to replace the first two terms involving $\breve{w}_a{}^i$ in the third line back by $\partial_t\breve{\kappa}_a{}^i$. One then obtains,

$$\partial_t[\partial_n(q_{ij}\breve{\kappa}_n{}^j)] = \partial_t[q_i{}^j\partial_j\breve{\kappa}_n{}^n - k_i{}^j\breve{\kappa}_j{}^n] + \partial_n[2q_i{}^j n^k\breve{w}_{[kj]}(0)], \tag{3.58}$$

therefore, integrating in time one finally obtains

$$\begin{aligned}\partial_n[q_{ij}\breve{\kappa}_n{}^j(t)] - \partial_n[q_{ij}\breve{\kappa}_n{}^j(0)] = {}& [q_i{}^j\partial_j\breve{\kappa}_n{}^n(t) - k_i{}^j\breve{\kappa}_j{}^n(t)]\\ & -[q_i{}^j\partial_j\breve{\kappa}_n{}^n(0) - k_i{}^j\breve{\kappa}_j{}^n(0)]\\ & +t\partial_n[2q_i{}^j n^k\breve{w}_{[kj]}(0)],\end{aligned} \tag{3.59}$$

which is Eq. (3.47). The Eq. (3.48) is obtained from the equation

$$\breve{\rho}(t) + \rho_0\breve{\kappa}_T(t) = \breve{\rho}(0) + \rho_0\breve{\kappa}_T(0), \tag{3.60}$$

with $\breve{\kappa}_T$ the trace of $\breve{\kappa}_a{}^i$, that is, $\breve{\kappa}_T = \delta_i{}^a\breve{\kappa}_a{}^i$. The Eq. (3.60) is a time integration of the trace of the evolution Eq. (3.8) after replacing $\breve{w}$ with the evolution Eq. (3.9). It is the linearization of the equation $\rho(t)\kappa(t) = \rho(0)\kappa(0)$, which can be obtained from the nonlinear Eqs. (2.19), (2.20), where $\kappa = \det(\kappa_a{}^i)$. Take the normal derivative of Eq. (3.60),

$$\partial_n\breve{\kappa}_n{}^n = -\partial_n\left[q_i{}^a\breve{\kappa}_a{}^i + \frac{\breve{\rho}(t)}{\rho_0}\right] + \partial_n\left[\breve{\kappa}_T(0) + \frac{\breve{\rho}(0)}{\rho_0}\right], \tag{3.61}$$

and in the right hand side replace the first two terms using Eq. (3.46) and (3.45), respectively. The result is Eq. (3.48).

The normal derivatives of the acceleration are obtained as follows. The Eq. (3.49) is the contraction of the constraint Eq. (3.20) with the normal vector n^a. The Eq. (3.50) is just the evolution Eq. (3.12).

The normal derivatives of $\breve{w}_{ij}$ are found as follows. The Eq. (3.51) is the contraction of the evolution Eq. (3.11) with the normal vector n^i. The Eq. (3.52) is the normal derivative of the time integral of the evolution Eq. (3.13). The Eq. (3.53) is the contraction of the constraint Eq. (3.19) with the normal vector n^a. The last Eq. (3.54) is obtained using, once more, that the vorticity $\breve{w}_{[ij]}$ is conserved in time. Start form the identity

$$\breve{w}_{jk} = \breve{w}_{kj} + 2\breve{w}_{[jk]}, \tag{3.62}$$

compute the component $n^j q_i{}^k$ and the take a normal derivative. The result is

$$\begin{aligned}\partial_n(\breve{w}_{nk}q_i{}^k) &= \partial_n(q_i{}^j\breve{w}_{jn}) + \partial_n[2q_i{}^j n^k\breve{w}_{[jk]}(t)], && (3.63)\\ &= q_i{}^j\partial_j\breve{w}_{nn} - k_i{}^j\breve{w}_{jn} - k_i{}^j\breve{w}_{nj} + \partial_n[2q_i{}^j n^k\breve{w}_{[jk]}(0)], && (3.64)\end{aligned}$$

where in the second line we used Eq. (3.53) contracted with n^k. Replace $\breve{w}_{nn}$ using $\breve{w}_{nn} = \breve{w} - q^{ij}\breve{w}_{ij}$, and the result is Eq. (3.54).

Finally note that Eqs.(3.51)–(3.54) imply that $\partial_n\breve{w}_{nn}$ can be expressed in terms of normal derivatives of $\breve{w}$ and $q^{ij}\breve{w}_{ij}$, and these, in turn, by tangential derivatives of the rest of the fields using Eqs. (3.51), (3.53). $\square$

4. Discussion. We have presented a new method for treating initial free–boundary value problems. That method consists in transforming the fluid equations to Lagrangian coordinates and then enlarging the system by adding every first derivative of the velocity field as new unknowns. Such a system, which we call boundary adapted, consists of evolution and constraint equation.

In the case treated here we used the fact that the linearization of this system near a static solution has a particular form. The system decouples in two subsystems: the first one consists of the unknowns $\check{a}_i$, $\check{w}$ and of the Eqs. (3.11), (3.12); the second one consists of the rest of the evolution equations. The first subsystem is symmetric hyperbolic and can be solved independently of the second. This second subsystem is formed by ordinary differential equations in time with the solution of the first subsystem as source functions. This decomposition of the linearised equations holds for static background solutions, but not for more general background solutions.

The linear stability of the initial free–boundary compressible Euler equations has already been shown in [9]. Gravitational effects were neglected. The proof is not based on the method of symmetric hyperbolic systems with maximally dissipative boundary condition. It is based, instead, on techniques specific to the equations under investigation. The proof start by transforming Euler equations to Lagrange coordinates. The problem to solve is then an initial fixed-boundary value problem for the space position vector field. Linear stability for this problem is proved by linearising the equations for this unknown vector field. One can show that the linear equation is weakly hyperbolic, with the speed of sound being the nonzero characteristic speed. The crucial step is to check that a pseudodifferential first order reduction is weakly hyperbolic. (See [11] and references therein for a definition of first order pseudodifferential reductions.) Therefore, special techniques have been developed in [9], [10] [3] to solve this linear problem. These are based in a decomposition of the unknown vector field into a divergence part and a divergence-free part. The equation decouples to highest order in the linearization parameter, and different techniques are used to estimate each part of the equation. The well posedness for the linear problem then follows from these estimates.

A condition for the background solution is assumed in [9], that prevents the Rayleigh-Taylor instabilities in incompressible fluids, namely $\partial_n p \leq -c_0 < 0$, in ∂D (Eq. (1.9) in that reference). This instability appears in incompressible fluids, essentially when a heavier fluid is on top of a lighter one in a constant gravitational field [18],[1]. One can check, however, that the argument to show this instability does not hold for compressible fluids. (For example, reproduce for a compressible fluid the proof given in [2].) So it is not clear whether this instability appears in a compressible fluid. The only previous proof of well posedness for the linearization of the initial-free boundary value problem in compressible fluids is the one given in [9]. And that proof makes central use of a condition in the background solution that prevents this instability. In our case, the static self-gravitating fluid satisfies this condition. However Theorem 3.2 can be generalised to the case where self-gravitational effects are discarded, that is, $\phi = 0$. We still need the condition that the background solution be static. This means that the mass density must be constant everywhere in the background solution. And that this constant must be the one that makes the pressure vanish. Therefore, Theorem 3.2 can be generalised to a background solution that represents dust, and which does not satisfy the condition to prevent Rayleigh-Taylor instability in incompressible fluids.

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Appendix A. Background concepts on hyperbolic equations.

We present the notation, the definitions, and the main results from the literature needed for our problem. We start in Sec. A.1 defining symmetric, weakly and strongly hyperbolic systems [7]. We then briefly review that the Cauchy problem for linear, variable coefficients, weakly hyperbolic systems is not well posed. In Sec. A.2 we summarized the initial fixed-boundary problem for symmetric hyperbolic systems. It is known that under certain assumptions this problem is well posed. We review a well posedness theorem for linear systems given in [16], which is the main result from the literature needed for this work.

A.1. Symmetric, weakly, and strongly hyperbolic systems. Let $D \subset \mathbb{R}^n$, with $n \geq 1$, be a compact set, lying on one side of its C^∞ boundary ∂D. Let $D_T = [0,T] \times D$, $B_T = [0,T] \times \partial D$. Consider the quasilinear first order operator

$$L = A^0 \partial_t + \sum_{j=1}^{n} A^j \partial_j + B, \tag{A.1}$$

where $A^0, A^1, \ldots, A^n$, B are given real $N \times N$ matrix valued functions of (t,x,u), with $(t,x) \in D_T$, and $u \in \mathbb{R}^N$. The operator is called linear if none of the matrices depend on u.

Definition A.1. *The operator (A.1) is called symmetric hyperbolic if the matrices $A^0, \ldots, A^n$ are real symmetric in $D_T \times \mathbb{R}^N$ and there exists a constant $0 < a_0$ such that $\mathbb{I} a_0 \leq A^0(t,x) \leq \mathbb{I}\left(a_0^{-1}\right)$ for all $(t,x) \in D_T$ and $u \in \mathbb{R}^N$, where $\mathbb{I}$ is the identity $N \times N$ matrix.*

Definition A.2. *The quasilinear operator (A.1) is called weakly hyperbolic if the matrix $P(\omega) := (A^0)^{(-1)} A^j \omega_j$ has real eigenvalues, for every $\omega \in \mathbb{R}^n$, $(t,x) \in D_T$, and $u \in \mathbb{R}^N$.*

This is the definition given in [7] Page 57, for the case of linear system with constant coefficients. No definition is given in that reference for variable coefficient systems, because the Cauchy problem is not well posed in this case. This is shown by examples in Sec. 2.2.3 and 2.2.4 in that reference. Well posedness requires extra assumptions on the system.

For example, assume that system (A.1) is symmetric hyperbolic. Therefore, matrices $A^1, \ldots, A^n$ are all symmetric, and then $P(i\omega)$ is symmetric for every $\omega \in \mathbb{R}^n$, which implies that it is diagonalizable. The matrix $P(\omega)$ not only has real eigenvalues, but also has a complete set of eigenvectors. It turns out that this second property is critical to prove well posedness of the Cauchy problem.

This is the reason for the definition of a class of systems wider than symmetric hyperbolic, but narrower than weakly hyperbolic, which have a well posed Cauchy problem.

Definition A.3. *The operator (A.1) is called strongly hyperbolic if the matrix $P(\omega) := (A^0)^{(-1)} A^j \omega_j$ has real eigenvalues at every $(t,x) \in \overline{D}_T$, and a complete set of linearly independent eigenvectors for every $\omega \in \mathbb{R}^n$, which depend smoothly on t, x, u, and ω.*

For linear operators this is essentially the definition given in [7] page 186, if one recalls Lemma 2.4.2 in that reference. The Cauchy problem for strongly hyperbolic systems is well posed. See Theorem 6.2.2 in [7] for linear operators, and Theorem 5.2.D in [19] for quasilinear operators.

A.2. Results on fixed-boundary. Consider the initial fixed-boundary value problem for a linear symmetric hyperbolic operator L, given by

$$Lu = F \quad \text{in} \quad D_T, \tag{A.2}$$
$$Mu = 0 \quad \text{on} \quad B_T, \tag{A.3}$$
$$u|_{t=0} = f \quad \text{in} \quad D. \tag{A.4}$$

The function u, and the given functions F and f are vector valued functions with N components. Functions u and F are defined on D_T, while f is defined on D. M is a given real $d \times N$ matrix defined on ∂D, and it has constant rank d everywhere on ∂D.

Well posedness for a Cauchy problem or an initial boundary value problem essentially means that there exists a unique solution for some given initial and boundary data, and this solution depends continuously on the data.

Definition A.4. *Let $B(D)$ be a Banach space with norm $|\ |$, whose elements are vector valued functions $u : D \to \mathbb{R}^N$. The initial boundary value problem (A.2)–(A.4) is well posed in a Banach space $B(D)$ if given initial data $f \in B(D)$ and appropriate boundary data M, there exists a solution $u(t,x)$, which is unique in $B(D)$ for each $t \in [0,T)$; and satisfies the estimate*

$$|u(t)|^2 \leq C_1 e^{C_2 t} \left(|f|^2 + C_3 \int_0^t |F(t')|^2 dt' \right), \tag{A.5}$$

for some positive constants C_1, C_2, C_3.

Let $n = (n_1, \dots, n_n)$ be the unit outward normal to ∂D. Define the normal matrix A_n as

$$A_n = \sum_{j=1}^{n} A^j n_j. \tag{A.6}$$

The boundary is called non-characteristic if A_n is invertible everywhere on ∂D. On the other hand, if the matrix A_n is not invertible but has constant rank on ∂D then the boundary is said to be characteristic of constant multiplicity. *The boundary condition (A.3) is called maximal dissipative (or positive) if $M = M(t,x) \in C^\infty(\partial D)$ is a real $d \times N$ matrix valued function of constant rank d on ∂D and* $\ker M$ *is maximal positive for A_n.* The condition that $\ker M$ is maximal positive for A_n means that A_n is positive definite for every vector in $\ker M$, and $\ker M$ is the biggest space with this property, that is if $\langle v, A_n v\rangle > 0$ then $v \in \ker M$. Here $\langle\ ,\ \rangle$ is the product in $\mathbb{C}^n$.

The initial and boundary data overlap at ∂D. *The compatibility condition of order $p \geq 0$ is given by $Mf^{(k)} = 0$ on ∂D, for $k = 0, \dots, p$.* Here the $f^{(k)}$ are defined recursively starting from the initial data. First, $f^{(0)} = f$, and the $f^{(k)}$, $k \geq 1$ are computed by formally taking the $k-1$ time derivative of $Lu = F$, then solving for $\partial_t^k u$, and evaluating it at $t = 0$.

It is proved in [16] that the initial fixed-boundary value problem (A.2)–(A.4) is well posed in the Hilbert spaces, $H_*^m(D)$. These spaces are a generalization of the usual Sobolev spaces $H^m(D)$. Functions in $H_*^m(D)$ satisfy that at the boundary ∂D they are twice more differentiable in directions tangential to ∂D than in the normal direction.

The definition of spaces $H_*^m(D)$ requires the concept of vector fields tangential and normal to ∂D. The vector field $\tau \in C^\infty(D; \mathbb{R}^n)$ is called tangential if and only if $\langle \tau, \nu \rangle = 0$ at every point in ∂D. A vector field $\pi \in C^{\infty}(D; \mathbb{C}^n)$ is called normal if $\pi = \nu$ at every point ∂D. Let $\{\pi, \tau_2, \ldots, \tau_n\}$ be a set of smooth vector fields on D, linearly independent at each point in D, such that π is a normal vector and $\tau_2, \ldots, \tau_n$ are tangential vectors. Let $L^2(D)$ be the space of square integrable functions in D, and denote by $\| \ \|$ its norm. Given $m \geq 1$ the function space $H^m_*(D)$ is defined as the set of functions $u \in L^2(D)$ such that

$$\|u\|^2_{m,*} = \sum_{|\alpha|+2k \leq m} \|\pi^k \tau^{\alpha}(u)\|^2. \tag{A.7}$$

Here $\alpha = (\alpha_2, \ldots, \alpha_n)$ is a multi-index, $|\alpha| = \alpha_2 + \ldots + \alpha_n$ and $\tau^{\alpha} = \tau_2^{\alpha_2} \ldots \tau^{\alpha_n}$.

Let X be a Banach space and $T > 0$, then $C^k([0,T]; X)$ denote the space of k-times continuously differentiable functions defined on $[0,T]$ taking values in X. We define

$$C_T(H^m_*) = \bigcap_{k=0}^{m} C^k([0,T]; H^{m-k}_*(D)), \tag{A.8}$$

with the norm $|||u|||_{m,*,T} = \sup_{[0,T]} |||u(t)|||_{m,*}$, where

$$|||u(t)|||_{m,*} = \sum_{k=0}^{m} \|\partial_t^k u(t)\|^2_{m-k,*}. \tag{A.9}$$

Finally define the norm $|||f|||_{m,*}$ for the initial data function f as follows,

$$|||f|||^2_{m,*} = \sum_{k=0}^{m} \|f^{(k)}\|^2_{m-k,*}, \tag{A.10}$$

where $f^{(k)}$ are the functions that enter into the compatibility conditions at ∂D.

The result needed in this work is the well posedness of the initial fixed-boundary value problem for symmetric hyperbolic equations. We reproduce here a weaker version of the theorem proved in [16]. The most general result can be found in that reference.

Theorem A.5. *Consider the initial fixed-boundary value problem (A.2)–(A.4) for the symmetric hyperbolic operator L. Let s, p be integers such that $s \geq 2[\frac{3}{2}] + 6$ and $1 \leq p \leq s$. Assume the following:*

1. *The matrices $A_0, \ldots, A_n$, B and the source function F belong to $C_T(H^p_*)$ for $p \leq s$.*
2. *$d = \mathrm{rank}(A_n)$ is constant for every point in B_T, and $0 < d < N$.*
3. *The boundary condition (A.3) is maximal dissipative.*
4. *The initial data f satisfies the compatibility condition of order $s - 1$ at ∂D. It also satisfies that $f^{(p)} \in H^{s-p}_*(D)$ for $p = 0, \ldots, s$.*

Then there exists a unique solution $u \in C_T(H^s_*)$ of the initial fixed-boundary problem (A.2)–(A.4). Moreover, there exist positive constants C_1, C_2, C_3, such that

$$|||u(t)|||^2_{p,*} \leq C_1 e^{C_2 t} \left(|||f|||^2_{p,*} + C_3 \int_0^t |||F(t')|||^2_{p,*} dt' \right) \tag{A.11}$$

for each $t \in [0,T]$.

Appendix B. Symmetric hyperbolic Lagrange formulation.

We present an alternative form of the Lagrange formulation of Euler equations. This system has symmetric hyperbolic evolution equations in space dimension greater or equal 1. The idea is to add appropriate combinations of the constraint equations (2.25)–(2.30) to the evolution equations (2.18)–(2.23).

The symmetric hyperbolic Lagrange formulation of Euler equations is the following. Consider the domain D_T, and coordinates (t, y^a), where the coordinates y^a represent the fluid particles. The unknown fields are $(\hat{x}^i,\ {\kappa_a}^i,\ \hat{\phi},\ \hat{\rho},\ \hat{v}^i,\ \hat{a}_i,\ \hat{w}_{ij})$, representing the same fields as in Sec. 2.2. The evolution equations are

$$\partial_t \hat{x}^i = \hat{v}^i, \tag{B.1}$$

$$\partial_t {\kappa_a}^i = {\kappa_a}^j {\hat{w}_j}^{\ i}, \tag{B.2}$$

$$\partial_t \hat{\rho} = -\hat{\rho}\, {\hat{w}_i}^{\ i}, \tag{B.3}$$

$$\partial_t \hat{v}^i = \hat{a}^i, \tag{B.4}$$

$$\partial_t \hat{a}_i - \hat{\nu}^2 \hat{\partial}_j {\hat{w}_i}^{\ j} = -\hat{\alpha}\hat{w}(\hat{a}_i + \hat{\partial}_i\hat{\phi}) - {\hat{w}_i}^{\ j}\hat{a}_j - \hat{\partial}_i(\partial_t\hat{\phi}), \tag{B.5}$$

$$\hat{\nu}^2 \partial_t \hat{w}_{ij} - \hat{\nu}^2 \hat{\partial}_j \hat{a}_i = -\hat{\nu}^2 {\hat{w}_i}^{\ k}\hat{w}_{kj}. \tag{B.6}$$

The gravitational potential is given by Eq. (2.24). The constraint equations are given by (2.25)–(2.30).

This is the symmetric hyperbolic Lagrange formulation. The only difference with respect to the boundary adapted system are Eqs. (B.5), (B.6). The first one comes from adding the constraint $\hat{\nu}^2(\hat{\partial}_j {\hat{w}_i}^{\ j} - \hat{\partial}_i {\hat{w}_j}^{\ j}) = 0$ to Eq. (2.22). The second one is obtained as follows. Add the constraint $\hat{\partial}_{[i}\hat{a}_{j]} = 0$ to Eq. (2.23), and multiply the result by $\hat{\nu}^2$. This finishes the procedure to obtain Eqs. (B.1)–(B.6).

Lemma B.1. *The evolution equations (B.1)–(B.6) are symmetric hyperbolic.*

The proof is a straightforward computation from the principal part of equations (B.1)–(B.6), and is not reproduced here.

The symmetric hyperbolic Lagrange formulation (B.1)–(B.6), (2.25)–(2.30) can be translated back to the Euler formulation. The result is the following system. Let (t, x^i) be coordinates in $\mathbb{R}^4$. The dynamical variables are $(\rho,\ v^i,\ a^i,\ w_{ij})$, that is the fluid mass density, the fluid velocity, the material acceleration, and the space derivatives of the fluid velocity, respectively. The equations consist of evolution and constraint equations. The evolution equations are given by

$$D_t \rho = -\rho w, \tag{B.7}$$

$$D_t v^i = a^i, \tag{B.8}$$

$$D_t a_i - \nu^2 \partial_j {w_i}^j = -\alpha w(a_i + \partial_i\phi) - {w_i}^j a_j - \partial_i(D_t\phi), \tag{B.9}$$

$$\nu^2 D_t w_{ij} - \nu^2 \partial_j a_i = -\nu^2 {w_i}^k w_{kj}, \tag{B.10}$$

where $D_t = \partial_t + v^i\partial_i$ is the material derivative, the gravitational potential ϕ is given by Eq. (1.8), $\nu^2 = \frac{(\partial p)}{(\partial \rho)} q$ is the square of the sound velocity, $\alpha = \frac{\rho\beta}{(\nu^2)}$ and $\beta = \frac{\partial^2 p}{\partial \rho^2}$. We use the notation $w = {w_i}^i$, and Latin indices i, j, k, l are rised and lowered with δ^{ij}, and δ_{ij}, respectively. The constraint equations are the following,

$$\partial_i v^j = {w_i}^j, \quad \text{(B.11)} \qquad \partial_{[i} {w_{j]}}^k = 0, \quad \text{(B.13)}$$

$$\frac{\partial_i p}{\rho} + \partial_i\phi = -a_i, \quad \text{(B.12)} \qquad \partial_{[i} a_{j]} = 0. \quad \text{(B.14)}$$

Both systems, however, are not well adapted to study well posedness of initial free–boundary problems. This means that it is not clear how to prescribe boundary data satisfying the three conditions given in Theorem 3.2. That is, the boundary data being maximal dissipative for the evolution equations, it implies the constraint preservation, and the resulting solution has a free–boundary.

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