AMCS 610

Problem set 9 due April 29, 2014

Dr. Epstein

Reading: Read Chapters 23, 27, and 28 in Lax, *Functional Analysis*.

Standard problem: The following problems should be done, but do not have to be handed in.

1. Lax page 244, exercise 10.
2. Lax page 241, exercise 5.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Let H be a Hilbert space, and $\{y_n\}$ a set of unit vectors satisfying the hypotheses of Theorem 7 in §22.5.

   (a) Show that there is a constant C so that, for all $x \in H$, we have:

   $$\frac{1}{C}\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, y_n\rangle|^2 \leq C\|x\|^2 \tag{1}$$

   (b) We can normalize the $\{y_n\}$ so that $\langle x_n, y_n\rangle$ is real, for all n, and define $\{\theta_n\}$ so that $\cos\theta_n = \langle x_n, y_n\rangle$, and $\theta_n \in [0, \pi)$. Show that

   $$\sum_{n=1}^{\infty} |\theta_n|^2 < \infty. \tag{2}$$

   (c) For $k \in \mathbb{N}$ set $n_k = k + \epsilon_k$; find a non-trivial condition on $\{\epsilon_k\}$, so that the set of functions $\{\sin n_k x : k \in \mathbb{N}\}$ defines a basis for $L^2([0, \pi])$.

2. Define

   $$Kf(x) = \int_0^{\pi} \log|x - y| f(y) dy. \tag{3}$$

   Show that $K : L^2[0, \pi] \to L^2[0, \pi]$ and $K : C^0[0, \pi] \to C^0[0, \pi]$ are both compact operators.

3. Suppose that we define the projection operator on $L^2(S^1)$, in terms of the Fourier representation by

$$P\left(\sum_{j=-\infty}^{\infty} a_j e^{ij\theta}\right) = \sum_{j=0}^{\infty} a_j e^{ij\theta}. \tag{4}$$

(a) Show that P is self adjoint.

(b) Let $g \in C^0(S^1; \mathbb{C})$, define the multiplication operator $M_g f = gf$. Show that the composition satisfies:

$$\|M_g f\|_{L^2} \leq \|g\|_{L^\infty} \|f\|_{L^2}. \tag{5}$$

What is the adjoint of PM_g?

(c) Let g be a trigonometric polynomial:

$$g = \sum_{j=-N}^{N} b_j e^{ij\theta} \tag{6}$$

Show that the commutator, $[P, M_g] = PM_g - M_g P$, is a finite rank operator.

(d) Show that if $g \in C^0(S^1; \mathbb{C})$, then the commutator $[P, M_g]$ is a compact operator. Hint: Approximate g.

(e) If we let $H^2(S^1)$ denote the range of P, then we see that $PM_g : H^2(S^1) \to H^2(S^1)$. We sometimes write this operator as $PM_g P$. If $g_1, g_2 \in C^0(S^1; \mathbb{C})$, then prove that

$$PM_{g_1} P P M_{g_2} P - PM_{g_1} M_{g_2} P \tag{7}$$

is a compact operator.

(f) Show that if g is a non-vanishing continuous function on S^1, then there is an operator $A : H^2(S^1) \to H^2(S^1)$ so that

$$PM_g PA - \mathrm{Id} \text{ and } APM_g P - \mathrm{Id} \tag{8}$$

are compact operators.

(g) Show that $PM_g P : H^2(S^1) \to H^2(S^1)x$ has a closed range of finite codimension.

(h) Find non-vanishing, continuous functions g_1, g_2 on S^1 so that $\ker PM_{g_1} P$ is non-trivial, and the range of $PM_{g_2} P$ is not all of $H^2(S^1)$.