AMCS 610
Problem set 8 due April 22, 2014
Dr. Epstein

Reading: Read Chapters 17, 21, and 22, in Lax, *Functional Analysis*.
Standard problem: The following problems should be done, but do not have to be handed in.

1. Lax page 201, exercise 1.
2. Lax page 201, exercise 2.
3. Lax page 201, exercise 4.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Suppose that X, U are Banach spaces and $C : X \to U$ is compact. Prove that if $< x_n >$ is sequence in X, which converges weakly to x, then $< Cx_n >$ converges in norm to Cx.
2. Prove that the two definitions of precompact set given on page 233 of Lax are equivalent.
3. Let X be a Banach space, and $K \subset X$ a precompact subset of X. Show that the convex hull of K is also precompact.
4. Let X be a Banach space and $\{P_N : N \in \mathbb{N}\}$ be a sequence of finite rank operators, which converge strongly to the identity, that is $\lim_{N\to\infty} P_N x = x$, for every $x \in X$. If $C : X \to X$ is a compact operator, then prove that $P_N C$ converges to C in the uniform norm. Show that if X is a Hilbert space, then any compact map $C : X \to X$ is the norm limit of a sequence of finite rank maps.
5. Suppose that X is a Hilbert space and $C : X \to X$ is a compact *self adjoint* operator, that is $\langle Cx, y\rangle = \langle x, Cy\rangle$, for all $x, y \in X$.

   (a) Prove that for all $x \in X$, the function $F(x) = \langle Cx, x\rangle$ is real valued.

(b) Suppose that for some x, $F(x) > 0$; show that there is unit vector $x_1 \in X$, so that

$$F(x_1) = \sup\{F(x) : x \in X \text{ with } \|x\| = 1\}. \tag{1}$$

(c) Prove that x_1 is an eigenvector of C, that is, there is a real number λ_1 so that $Cx_1 = \lambda_1 x_1$. Give an example of a bounded (though non-compact), self adjoint operator A on a Hilbert space for which this is **not** true.

(d) If we let $X_1 = \{x \in X : \langle x, x_1 \rangle = 0\}$, then C maps X_1 to itself, that is $CX_1 \subset X_1$.

6. Let $X = L^2([0, 1])$, and define the operator $Mf(x) = xf(x)$.

   (a) Prove that M is a bounded operator.

   (b) Does there exist a $\lambda \in \mathbb{C}$ and $f \in X$ such that $(M - \lambda \, \mathrm{Id})f = 0$?

   (c) What is the spectrum of M? Give a formula for the resolvent operator $R(\lambda) = (M - \lambda \, \mathrm{Id})^{-1}$. Where is it defined?

   (d) Suppose that $\varphi(z)$ is analytic on a neighborhood of the spectrum of M. Give the most explicit formula that you can for $\varphi(M)$. If $\varphi(z) = \sin(\pi z)$, then what is the spectrum of $\varphi(M)$?

7. Let $k(s, t)$ be a C^1-function on $[0, 1] \times [0, 1]$. Define the operator K by

$$Kf(s) = \int_0^1 k(s, t) f(t) dt. \tag{2}$$

   Show that $K : C^0([0, 1]) \to C^0([0, 1])$ and $K : L^2([0, 1]) \to L^2([0, 1])$ are compact operators.