AMCS 610
Problem set 7 due April 8, 2014
Dr. Epstein

Reading: Read Chapters 16 and 21 in Lax, *Functional Analysis*.
Standard problem: The following problems should be done, but do not have to be handed in.

1. Lax page 182, exercise 1.
2. Lax page 184, exercise 3.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. For $\alpha > 0$, define an integral operator
$$K_\alpha f(x) = \int_0^x \frac{f(y)dy}{(x-y)^{1-\alpha}}. \tag{1}$$
   (a) Show that K_α defines a continuous map from $C^0[0,1]$ to itself.
   (b) Show that K_α defines a continuous map of $L^p[0,1]$ to itself for every $1 \leq p < \infty$.
   (c) Define the function
$$B(\alpha,\beta) = \int_0^1 \frac{du}{(1-u)^{1-\alpha}u^{1-\beta}}. \tag{2}$$
   Show that $K_\alpha \circ K_\beta = B(\alpha,\beta)K_{\alpha+\beta}$.
   (d) What happens in the previous part if $0 < \alpha < 1$, and we set $\beta = 1-\alpha$? Use this to find a formula for K_α^{-1}. Can you give a class of functions for which the formula for K_α^{-1} makes sense?

2. Recall that we define the Fourier transform on $L^2(\mathbb{R})$, by first defining it via an integral for $f \in C_c^0(\mathbb{R})$, then using the density of $C_c^0(\mathbb{R})$ in $L^2(\mathbb{R})$, and the Parseval formula
$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi \tag{3}$$

to extend it as a bounded map from $L^2(\mathbb{R})$ to itself. Let $f \in L^2(\mathbb{R})$; for $R > 0$ define

$$\hat{f}_R(\xi) = \int_{-R}^{R} f(x)e^{-ix\xi}dx. \tag{4}$$

(a) Prove that $\underset{R\to\infty}{\text{l. i. m.}}\, \hat{f}_R = \hat{f}$, that is $< \hat{f}_R >$ converges in the L^2-sense to $\hat{f}$.

(b) Suppose that for any $\delta > 0$, the functions $< \hat{f}_R(\xi) >$ converge uniformly to $g(\xi)$ for $\delta \leq |\xi| \leq \delta^{-1}$. Prove that $g = \hat{f}$ almost everywhere.

(c) Prove that if $f(x) = (x + i\epsilon)^{-1}$ for an $\epsilon > 0$, then

$$\hat{f}(\xi) = -2\pi i \chi_{[0,\infty)}(\xi)e^{-\epsilon\xi}. \tag{5}$$

(d) Let $f_\epsilon(x) = x^{-1}\chi_{[\epsilon,\infty)}(|x|)$; for $\epsilon > 0$, these functions belong to $L^2(\mathbb{R})$. Compute $\hat{f}_\epsilon(\xi)$. Show that the operators $H_\epsilon g = g \mapsto f_\epsilon * g$ are bounded as maps from L^2 to itself, and

$$\underset{\epsilon\to 0^+}{\text{s-lim}} H_\epsilon = \mathcal{H}. \tag{6}$$

Here $\mathcal{H}$ is the Hilbert transform.

3. Let $k(s, t) \in C^0([0, 1] \times [0, 1])$, and, for $f \in C^0([0, 1])$ define

$$Kf(s) = \int_0^s k(s, t)f(t)dt. \tag{7}$$

(a) Prove that $K : C^0([0, 1]) \to C^0([0, 1])$ is continuous.

(b) Let $M = \|k\|_\infty$. Show that, for $s \in [0, 1]$,

$$|Kf(s)| \leq Ms\|f\|_\infty. \tag{8}$$

(c) For each $n \in \mathbb{N}$ show that, for $s \in [0, 1]$,

$$|K^n f(s)| \leq \frac{(Ms)^n}{n!}\|f\|_\infty. \tag{9}$$

(d) Prove that $(\mathrm{Id} - K) : C^0([0, 1]) \to C^0([0, 1])$ is invertible and the inverse is given by the **norm** convergent series:

$$(\mathrm{Id} - K)^{-1} = \sum_{n=0}^{\infty} K^n. \tag{10}$$

Here the "norm" refers to the operator norm on $\mathscr{L}(C^0([0, 1]), C^0([0, 1]))$.

(e) Show that $(\lambda\,\mathrm{Id} - K)^{-1}$ exists for any $\lambda \in \mathbb{C} \setminus \{0\}$. Give examples of finite dimensional linear transformations (one for each dimension $n > 1$) that also have this property.

4. The Laplace transform is defined for absolutely integrable functions and $t \in (0, \infty)$ by the integral:

$$\mathscr{L} f(t) = \int_0^\infty f(x) e^{-xt} dx. \tag{11}$$

(a) Show that the map $f(x) \mapsto e^{\frac{y}{2}} f(e^y)$ is an unitary isomorphism from $L^2([0, \infty))$ to $L^2(\mathbb{R})$.

(b) Show that if $s \in \mathbb{R}$, then

$$\mathscr{L}(x^{-\frac{1}{2}+is})(t) = \Gamma\left(\frac{1}{2} + is\right) t^{-\frac{1}{2}-is} \tag{12}$$

(c) Use (a) and (b) to show that $\mathscr{L}$ extends to define a bounded map from $L^2([0, \infty))$ to itself. What is $\|\mathscr{L}\|$? You cannot use the argument from the book!

(d) What is the kernel function of $\mathscr{L}^2$? What is the norm of $\mathscr{L}^2$ as a map from $L^2([0, \infty))$ to itself?

(e) Does $\mathscr{L}$ have a bounded inverse on $L^2([0, \infty))$?