AMCS 610
Problem set 6 due April 1, 2014
Dr. Epstein

Reading: Read Chapters 15 and 16 in Lax, *Functional Analysis*.
Standard problem: The following problem should be done, but do not have to be handed in.

1. Let X, Y, W be Banach spaces, with sequences $< S_n >\in \mathcal{L}(Y, W)$, $< T_n >\in \mathcal{L}(X, Y)$.
   (a) If $S_n \to S$ and $T_n \to T$ strongly, then $S_n T_n \to ST$ strongly.
   (b) Suppose that S_n converges weakly to S and T_n converges strongly to T; show that $S_n T_n$ converges weakly to ST.
   (c) Find examples of $< S_n >, < T_n >$ both of which converge weakly to zero, but such that $S_n T_n$ does not. Hint: Look at the shift operator on bi-infinite square summable sequences: $S(x_j) = (x_{j+1})$.
2. Show that if
$$\int_{-\infty}^{\infty} |\phi(x)| dx < \infty, \tag{1}$$
then the operator
$$K_\phi f(x) = \int_{-\infty}^{\infty} \phi(x - y) f(y) dy \tag{2}$$
is bounded from $L^2(\mathbb{R})$ to itself.
3. Lax page 165, exercise 3.
4. Lax page 166, exercise 7.
5. Lax page 168, exercise 9.
6. Lax page 172, exercise 13.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Prove that if $g \in L^1([-\pi, \pi])$, and we define

$$a_n = \int_{-\pi}^{\pi} g(x) e^{-inx} dx, \tag{3}$$

then $\lim_{n \to \pm\infty} a_n = 0$. Hint: Approximate g in the L^1-norm by functions for which this is obvious and then use a continuity estimate for the maps $g \mapsto a_n$, $n \in \mathbb{Z}$.

2. Let $M : X \to Y$ be linear. Show that the range of M is dense if and only if $\ker M' = \{0\}$.

3. Let X be a Banach space and $T \in \mathscr{L}(X, X)$ with $|T| < 1$.

   (a) Prove that $S_n = \sum_{j=0}^{n} T^j$ is a norm convergent sequence in $\mathscr{L}(X, X)$. Let S denote the limit.

   (b) Prove that for every $y \in X$ we have

   $$(\mathrm{Id} - T) S y = y \tag{4}$$

   and conclude that $(\mathrm{Id} - T)$ is boundedly invertible. Give a estimate for $|S|$ in terms of $|T|$.

   (c) Show that if $M \in \mathscr{L}(X, X)$ is invertible, then there is an $\epsilon > 0$ so that every $N \in \mathscr{L}(X, X)$ with $|M - N| < \epsilon$ is also invertible. Briefly, invertibility is an open property in the operator topology.

   (d) If $M \in \mathscr{L}(X, X)$, then we define the reolvent set of M to be

   $$\rho(M) = \{\lambda \in \mathbb{C} : (M - \lambda \, \mathrm{Id}) \text{ is invertible } \}. \tag{5}$$

   Prove that $\rho(M)$ is an open subset of $\mathbb{C}$.

4. Suppose that X, Y are Banach spaces and $M : X \to Y$ is a surjective, bounded linear map. Show that there is a constant $c > 0$, so that for every $y \in Y$, there exists an $x \in X$ with

$$Mx = y \text{ and } \|x\| < c\|y\|. \tag{6}$$

5. Let H be a Hilbert space with $\{u_n\}$ an orthonormal basis.

(a) Define $T_k : H \to H$ by

$$T_k(\sum_{j=1}^{\infty} a_j u_j) = a_k u_k. \tag{7}$$

Prove that T_k converges to 0 in the strong sense, but not in the operator norm.

(b) Define $S_k : H \to H$ by

$$S_k(\sum_{j=1}^{\infty} a_j u_j) = \sum_{j=1}^{\infty} a_j u_{j+k}. \tag{8}$$

Show that S_k converges to 0 in the weak sense, but not in the strong sense.

6. Let X be a separable Banach space with $\{x_n\}$ a countable dense subset of the unit ball. We define a map $T : \ell_1 \to X$, by setting:

$$T(\boldsymbol{a}) = \sum_{j=1}^{\infty} a_j x_j. \tag{9}$$

(a) Prove that T is bounded.

(b) Prove that T is surjective. Hint: you should find a direct argument.

(c) Show that X is isomorphic to a quotient space of ℓ_1.

7. Let $S \subset C^0([0, 1])$, which is closed with respect to the L^2-norm. This means that if $< f_n > \subset S$, and there is a function $f \in L^2[0, 1]$ such that $\|f_n - f\|_{L^2} \to 0$, then f can be represented by a function in S.

(a) Show that S is also closed as a subspace of C^0.

(b) Show that there is a constant M so that, for $f \in S$, we have

$$\|f\|_\infty < M\|f\|_2. \tag{10}$$

Hint: use the closed graph theorem.

(c) Show that for each $y \in [0, 1]$ there is a function $k_y \in L^2([0, 1])$ so that

$$f(y) = \int_0^1 f(x) k_y(x) dx. \tag{11}$$

8. (a) Suppose that X and Y are Banach spaces, and $D \subset X$ is a linear subspace, which may not be closed. Suppose that $T : D \to Y$ has a closed graph, and is 1-1 and onto. If D is not closed, then T need not be continuous. Prove, however, that $T^{-1} : Y \to X$ is continuous.

(b) Let X denote continuous functions on [0, 1] that vanish at 0; $Y = C^0([0, 1])$; and $D \subset X$, those functions with a continuous first derivative. Show that $Tf = \partial_x f$ has a closed graph, and is a 1-1, onto map from D to Y. What is T^{-1}? Give an *elementary* proof that it is bounded as a map from $Y \to X$.

9. Suppose that $k(s, t)$ is a measureable function on $S \times T$ such that

$$
\begin{aligned}
M_1 &= \sup_{s \in S} \int_T |k(s,t)| dn(t) < \infty \text{ and} \\
M_2 &= \sup_{t \in T} \int_S |k(s,t)| dm(s) < \infty.
\end{aligned}
\tag{12}
$$

Show that for every $1 < p < \infty$ the operator

$$
Kf(s) = \int_T k(s,t) f(t) dn(t) \tag{13}
$$

is bounded from $L^p(T; dn) \to L^p(S; dm)$ with $\|K\|_{L^p \to L^p} \leq M_1^{\frac{1}{q}} M_2^{\frac{1}{p}}$. Here $p^{-1} + q^{-1} = 1$.