AMCS 610

Problem set 5 due March 18, 2014

Dr. Epstein

Reading: Read Chapters 10 and 11 in Lax, *Functional Analysis*. You might also want to look at the sections in Royden, *Real Analysis*, or Rudin, *Real and Complex Analysis* on the Baire Category Theorem and the Uniform Boundedness Principle.
Standard problem: The following problems should be done, but do not have to be handed in.

1. Let (X, d) be a metric space and U, V open dense subsets of X. Show that $U \cap V$ is also dense.

2. Exercise 3 on page 101 of Lax.

3. Exercise 5 on page 104 of Lax.

4. Exercise 6 on page 106 of Lax.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Let X be a complete, countable metric space. Show that X has a discrete subset Y so that $\overline{Y} = X$. A subset Y is discrete if for each $y \in Y$ the set $\{y\}$ is open as a subset of X.

2. Let $\{q_n\}$ be an enumeration of the $\mathbb{Q} \cap [0, 1]$. For each m we let

$$U_m = \bigcup_{n=1}^{\infty} (q_n - \frac{1}{m2^n}, q_n + \frac{1}{m2^n}) \cap [0, 1]. \tag{1}$$

Is it true that

$$\bigcap_{m=1}^{\infty} U_m = \mathbb{Q} \cap [0, 1]? \tag{2}$$

Why or why not?

3. Prove: If $< x_n >$ is a sequence in ℓ_1 that converges weakly to 0, then

$$\lim_{n \to \infty} \|x_n\|_1 = 0, \tag{3}$$

that is: $< x_n >$ also converges strongly to zero. Hints: Argue by contradiction, choose an appropriate subsequence, and use the fact that $\ell_1^* = \ell_\infty$ is a very big vector space.

4. Suppose that $< b_j >$ is a sequence of real numbers so that, for every real sequence $< a_j >$, converging to zero, the limit

$$\ell(\boldsymbol{a}) = \lim_{N \to \infty} \sum_{j=1}^{N} a_j b_j \tag{4}$$

exists. Prove that

$$\sum_{j=1}^{\infty} |b_j| < \infty. \tag{5}$$

5. Let (a_{ij}) be an infinite matrix with complex entries, $1 \leq i, j < \infty$. Suppose that for every convergent sequence $< s_j >$, and $1 \leq i$, we define

$$\sigma_i = \lim_{N \to \infty} \sum_{j=1}^{N} a_{ij} s_j, \tag{6}$$

if the limit exists.

Show that these limits exist, for all convergent sequences $< s_j >$, and define a sequence $< \sigma_i >$, with the same limit, if and only if the following conditions hold:

(a)
$$\lim_{i \to \infty} a_{ij} = 0 \text{ for each } j.$$

(b)
$$\sup_{1 \leq i < \infty} \sum_{j=1}^{\infty} |a_{ij}| < \infty.$$

(c)
$$\lim_{i \to \infty} \sum_{j=1}^{\infty} a_{ij} = 1.$$

Give an example of such a matrix for which there exists a non-convergent sequence, $< s_j >$, so that σ_j exists for every $j \in \mathbb{N}$, and the sequence $< \sigma_j >$ is convergent.

6. Let $\{f_n\}$ be a sequence of continuous, real valued functions defined on $[0, 1]$, such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for every $x \in [0, 1]$.

   (a) Prove that there is a non-empty open set $V \subset [0, 1]$, and a number M such that
   $$|f_n(x)| < M \text{ for all } x \in V. \tag{7}$$

   (b) If $\epsilon > 0$, show that there is a nonempty open set, V and an integer N so that if $n \geq N$, then
   $$|f(x) - f_n(x)| < \epsilon \text{ for all } x \in V. \tag{8}$$

   Hint: For each N define $A_N = \{x : |f_n(x) - f_m(x)| \leq \epsilon \text{ if } N \leq n, m\}$, and consider $\cup_N A_N$.

7. If $1 \leq p < q < \infty$, then $\ell_p \subset \ell_q$. For fixed $p < q$, and $n \in \mathbb{N}$, show that the set
$$B_n = \{(x_j) \in \ell_q : \sum_{j=1}^{\infty} |x_j|^p \leq n\} \tag{9}$$
is closed and nowhere dense, as a subset of ℓ_q. Hence, as a subset of ℓ_q, ℓ_p is a set of first category.