AMCS 610
Problem set 4
Due March 4, 2014
Dr. Epstein

**Reading:** Read Chapters 8.1-3, 9.1, and 10.1-3 in Lax, *Functional Analysis*.
**Standard problems:** The following problems should be done, but does not have to be handed in.

1. Exercise 1 on page 76 of Lax.
2. Prove that $L^2([0,1])$, the closure of $C^0([0,1])$ with respect to

$$\|f\|_2^2 = \int_0^1 |f(x)|^2 dx, \tag{1}$$

is a separable space.

**Homework assignment:** The solutions to the following problems should be carefully written up and handed in.

1. Let $\{w_1, \ldots, w_N\}$ be distinct points in the open unit disk, and $\{a_1, \ldots, a_N\}$ complex numbers. We define the affine subspace $Y_{\boldsymbol{a}} \subset \mathcal{H}^2(D_1)$, to be

$$Y_{\boldsymbol{a}} = \{f \in \mathcal{H}^2(D_1) : \; f(w_j) = a_j\}. \tag{2}$$

Show that $Y_{\boldsymbol{a}} \neq \emptyset$. Define

$$m(\boldsymbol{a}) = \inf\{\|f\|_2 : \; f \in Y_{\boldsymbol{a}}\}. \tag{3}$$

Prove that there is a unique function $f_1 \in Y_{\boldsymbol{a}}$ with $m(\boldsymbol{a}) = \|f_1\|$.

Let $g_w \in \mathcal{H}^2(D_1)$ be the unique function so that, for all $f \in \mathcal{H}^2(D_1)$,

$$f(w) = \int_{D_1} f(z)\overline{g_w(z)}dxdy \tag{4}$$

Prove that we can choose $(\lambda_1, \ldots, \lambda_N)$ so that

$$F_1 = \sum_{j=1}^{N} \lambda_j g_{w_j} \in Y_{\boldsymbol{a}}. \tag{5}$$

Let $Y_0$ be the subspace of $\mathscr{H}^2(D_1)$ consisting of functions that vanish at $\{w_j\}$. Prove $< F_1, F_0 >= 0$ for all $F_0 \in Y_0$ and explain why this shows that

$$m(\boldsymbol{a}) = \sqrt{\sum_{j,k=1}^{N} \langle g_{w_j}, g_{w_k} \rangle \lambda_j \overline{\lambda}_k} = \sqrt{\sum_{j=1}^{N} a_j \overline{\lambda}_j}. \tag{6}$$

Hence $F_1$ is the unique function in $Y_{\boldsymbol{a}}$ with

$$\|F_1\|_2 = m(\boldsymbol{a}). \tag{7}$$

2. Let $D$ be a connected open subset of $\mathbb{C}$. A function $f \in L^2(D)$ is weakly holomorphic if

$$\int_D f \partial_{\bar{z}} \varphi = 0, \tag{8}$$

for all $\varphi \in \mathscr{C}_c^\infty(D)$. Show that a weakly holomorphic function is smooth in the interior of $D$ and is a classical solution to the PDE $\partial_{\bar{z}} f = 0$. Note: Use the definition of weakly holomorphic given here, which differs from that given last semester. In particular, a weakly holomorphic function is not known, a priori, to be continuous.

3. The space $H_1(D_1)$ is the closure of $\mathscr{C}^\infty(\overline{D_1})$ with respect to the norm

$$\|u\|_1^2 = \int_{D_1} [|u(x)|^2 + |\nabla u(x)|^2] dx. \tag{9}$$

In class we proved that the map $R : u \mapsto u \restriction_{\partial D_1}$ has a continuous extension as a map $R : H_1(D_1) \to L^2(\partial D_1)$. Prove that there are functions $f \in L^2(\partial D_1)$ for which there does **not** exist a function $u \in H_1(D_1)$ for which $f = R(u)$. Hint: The argument given in class actually showed that $R(u)$ belongs to a subspace of $L^2(\partial D_1)$.

4. Review the definition of a *Banach Limit*, on pages 31-2 of Lax. As shown there, $\text{LIM}_{n\to\infty} a_n$ is a linear functional on bounded sequences with

$$\liminf_{n\to\infty} a_n \le \text{LIM}_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n. \tag{10}$$

Show that this implies that $\text{LIM}_{n\to\infty} \in \ell'_\infty$, but there does **not** exist a vector $\boldsymbol{b} = (b_1, b_2, \dots) \in \ell_1$, such that

$$\text{LIM}_{n\to\infty} a_n = \sum_{n=1}^{\infty} a_n b_n. \tag{11}$$

5. Exercise 2 on page 76 of Lax, and Exercise 3 on page 77 of Lax.

6. Exercise 5 on page 80 of Lax.

7. Suppose that $\ell$ is a bounded linear functional on a Hilbert space, and $\{e_j\}$ is a collection of orthonormal vectors. Show that

$$\lim_{j\to\infty} \ell(e_j) = 0. \tag{12}$$