AMCS 610

Problem set 3 due February 18, 2014

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Reading: Read Chapters 8.1-3, and 9.1 in Lax, *Functional Analysis*. **Standard problem:** The following problems should be done, but do not have to be handed in.

1. Exercises 1 and 2 on page 76 of Lax.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Let $\{V_1, \ldots, V_N\}$ be linearly independent vectors in a complex inner product space $(X, \langle\cdot,\cdot\rangle)$.

   (a) Show that the matrix
   $$G_{ij} = \langle V_i, V_j\rangle, \tag{1}$$
   is Hermitian symmetric that is $G_{ij} = \overline{G_{ji}}$.

   (b) Prove that for $(v_1, \ldots, v_n),\ (w_1, \ldots, w_N) \in \mathbb{C}^N$, we have
   $$\left|\sum_{i,j=1}^{N} G_{ij} v_i \overline{w}_j\right| \le \sqrt{\sum_{i,j=1}^{N} G_{ij} v_i \overline{v}_j}\sqrt{\sum_{i,j=1}^{N} G_{ij} w_i \overline{w}_j} \tag{2}$$

   (c) Without using computation, provide a conceptual proof that G_{ij} positive definite, i.e., there is a positive constant C so that for $(v_1, \ldots, v_n) \in \mathbb{C}^N$, we have
   $$\sum_{i,j=1}^{N} G_{ij} v_i \overline{v}_j \ge C \sum_{j=1}^{N} |v_j|^2. \tag{3}$$

   (d) Show that G_{ij} is invertible.

2. Suppose that $f \in C^0([0,1])$ and for every function $\varphi \in \mathscr{C}_0^\infty((0,1))$, we have
   $$\int_0^1 f(x)\varphi(x)dx = 0, \tag{4}$$

prove that $f = 0$ in $C^0([0,1])$. Now show that if $f \in L^2([0,1])$ and this condition holds for all $\varphi \in \mathscr{C}_0^\infty((0,1))$, then $f = 0$ in $L^2([0,1])$. Remember that $L^2([0,1])$ is the closure of $C^0([0,1])$ with respect to the L^2-norm. The proofs are completely different in the two cases!

3. Let $Y = \{u \in \mathscr{C}^\infty(\overline{D}_1) : \ \Delta u = 0\}$.

   (a) We let $\overline{Y}$ denote the closure of Y with respect to the L^2-norm on the unit disk:

   $$\|u\|_2^2 = \int_{D_1} |u(x,y)|^2 dx dy. \tag{5}$$

   Show that if $v \in \overline{Y}$ then v has representative that is smooth in the interior of the unit disk and that $\Delta v = 0$, in the interior of D_1. Hint: Use the Poisson formula.

   (b) Describe the radial functions in $Y^\perp$, that is functions of $r = \sqrt{x^2 + y^2}$ that are orthogonal to $\overline{Y}$.

4. A function u, which belongs to $L^2([-R, R])$ for all $R > 0$, is weakly constant if

   $$\int u(x) \partial_x \varphi(x) = 0 \tag{6}$$

   for every $\varphi \in \mathscr{C}_0^\infty(\mathbb{R})$. Show that a weakly constant function is smooth and constant, or more accurately: has a smooth representative, which is constant. Hint: Show that if $u(x)$ is weakly constant then so is $au(x - y)$ for all $a, y \in \mathbb{R}$.

5. For $f \in \mathscr{C}_0^\infty(\mathbb{R})$, show that $|f|$ has a weak derivative, which can be represented by a function $g(x)$ that satisfies

   $$|g(x)| \leq |\partial_x f(x)|. \tag{7}$$

   The statement that the weak derivative is *represented by* g means that for all $\varphi \in \mathscr{C}_0^\infty(\mathbb{R})$ we have

   $$\int_{\mathbb{R}} |f(x)| \partial_x \varphi(x) dx = -\int_{\mathbb{R}} g(x) \varphi(x) dx. \tag{8}$$

   Hint: Consider $\sqrt{f^2(x) + \epsilon^2}$. Compute the weak derivative of $|x|$.

6. Suppose that we define a weak solution of the wave equation,

$$\partial_x^2 u(x,t) - \partial_t^2 u(x,t) = 0, \tag{9}$$

to be a function that is square integrable in $[-R, R] \times [-R, R]$ for any R, and such that

$$\int_{\mathbb{R}^2} u(x,t)(\partial_x^2 \varphi(x,t) - \partial_t^2 \varphi(x,t))dxdt = 0, \tag{10}$$

for any function $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$. Show that if $f \in L^2(\mathbb{R})$, then

$$u(x,t) = f(x-t) \text{ and } v(x,t) = f(x+t) \tag{11}$$

are weak solutions of the wave equation. Hint: Approximate!