AMCS 610

Problem set 2 due February 11, 2014

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Reading: Read Chapters 3.2-3.3 (especially the proof of Theorem 8), 4.2, 5.1-5.2, 6.1-6.3 in Lax, *Functional Analysis*.

Standard problem: The following problems should be done, but do not have to be handed in.

1. Prove Theorem 4 in §3.2 of Lax.

2. Suppose that (X, d) is a metric space. Show that if $\lim_{n\to\infty} x_n = x^*$, then, for any $x \in X$, we also have that $\lim_{n\to\infty} d(x, x_n) = d(x, x^*)$.

3. For $1 \leq p \leq \infty$, prove that the normed vector space ℓ_p is a Banach space.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Prove that in any real normed linear space $(X, \|\cdot\|)$, the open and closed unit balls

$$B_1 = \{x \in X : \|x\| < 1\}, \quad \overline{B}_1 = \{x \in X : \|x\| \leq 1\} \tag{1}$$

are convex and have non-empty interior. The unit ball is *strictly convex*, if, whenever $\|x\| = \|y\| = 1$ and $x \neq y$, then

$$\left\| \frac{x+y}{2} \right\| < 1. \tag{2}$$

Show that the unit ball in ℓ_2 is strictly convex, but the unit ball in ℓ_1 is not.

2. A bounded sequence $< c_j >$ is Cesaro summable if

$$\lim_{n\to\infty} \frac{c_1 + \cdots + c_n}{n} \text{ exists.} \tag{3}$$

Show that a Banach limit LIM can be defined on ℓ_∞ so that if $< c_j >$ is Cesaro summable then

$$\underset{j\to\infty}{\text{LIM}} c_j = \lim_{n\to\infty} \frac{c_1 + \cdots + c_n}{n}. \tag{4}$$

3. Suppose that X is a Banach space and $Y \subset X$ is a closed subspace. Show that the quotient space X/Y, with the quotient norm

$$\|[x]\|_{X/Y} = \inf_{x \in [x]} \|x\|_X, \tag{5}$$

is complete.

4. Prove that every finite dimensional subspace of a normed vector space is closed. Hint: Use the fact that all norms on a finite dimensional vector space are equivalent to show that every finite dimensional subspace is complete.

5. Let $\mathcal{P}$ denote the subspace of $\mathcal{C}^0([0, 1])$ defined by polynomials restricted to $[0, 1]$. Suppose that $\ell : \mathcal{P} \to \mathbb{R}$ is a linear function with the property that

$$p(x) \geq 0 \text{ for } x \in [0, 1] \Rightarrow \ell(p) \geq 0. \tag{6}$$

Show that ℓ extends to define a linear functional , $\tilde{\ell}$, on all of $\mathcal{C}^0([0, 1])$, satisfying an estimate of the form

$$|\tilde{\ell}(f)| \leq C\|f\|_\infty. \tag{7}$$

Can you find a closed form expression for C?

6. Let $Y \subset \ell_\infty$ be the subspace of sequences that are eventually zero (only finitely many terms non-zero). Find the closure of Y with respect to the ℓ_∞-norm.

7. Prove that ℓ_1 has a countable dense subset, but ℓ_∞ does not.

8. [This problem assumes an elementary knowledge of holomorphic functions of one complex variable.] Let $\mathcal{H}^2(D_1)$ denote the closure of bounded holomorphic functions on the unit disk with respect to the L^2-norm

$$\|f\|_2^2 = \int_{D_1} |f(x, y)|^2 dxdy = \lim_{r \to 1^+} \iint_{D_r} |f(x, y)|^2 dxdy < \infty. \tag{8}$$

$L^2(D_1)$ is defined as the closure of $C^0(\overline{D}_1)$ with respect to the L^2-norm.

   (a) Show that $f \in \mathcal{H}^2(D_1)$ is holomorphic in $\operatorname{int} D_1$. That is, every element of $\mathcal{H}^2(D_1)$ has a representative that is holomorphic in $\operatorname{int} D_1$.

   (b) Show that if f is a square integrable function in D_1, which is holomorphic in the interior of D_1, then $f \in \mathcal{H}^2(D_1)$. (You need to show that f is an L^2-limit of functions in $C^0(\overline{D}_1)$.)

(c) Prove that for any $k \in \mathbb{N}$, there is a bounded linear functional ℓ_k defined on $L^2(D_1)$, so that if $f \in \mathcal{H}^2(D_1)$, then

$$\ell_k(f) = \partial_z^k f(0). \tag{9}$$