AMCS 610

Problem set 1 due February 4, 2014

Dr. Epstein

Reading: Read Chapters 1, 2, and 3 in Lax, *Functional Analysis*.

Standard problem: The following problems should be done, but do not have to be handed in.

1. Suppose that $K, L \subset X$, a real vector space, are convex sets. Prove that $K + L$ is also convex.

2. Let X, Y be real vector spaces and $M : X \to Y$ a linear map. Prove that if $K \subset X$ is convex, then $M(K)$ is convex, and if $L \subset Y$ is convex, then $M^{-1}(L)$ is convex.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. A linear function from a real vector space X to $\mathbb{R}$ is just a linear map $\ell : X \to \mathbb{R}$. Show that a linear function $\ell : \mathbb{R}^n \to \mathbb{R}$ is continuous with respect to the topology defined by any norm on $\mathbb{R}^n$.

2. Let X be a finite dimensional vector space, and $Y \subset X$ a proper subspace. Let $\{y_1, \ldots, y_k\}$ be a basis for Y. If $\dim X = n$, then show that there are vectors $\{x_1, \ldots, x_{n-k}\}$ so that $\{y_1, \ldots, y_k, x_1, \ldots, x_{n-k}\}$ is a basis for X. Conclude that
$$\dim X = \dim Y + \dim(X/Y). \tag{1}$$

3. Suppose that X is a finite dimensional real vector space.

   (a) Show that the set, X', of linear functions on X, with its natural vector space structure, has the same dimension as X. If $Y \subset X$ is a subspace, then the $\dim(X/Y)$ is called the *codimension* of Y, and
   $$Y^{\perp} = \{\ell \in X' : \ \ell(y) = 0 \text{ for all } y \in Y\}. \tag{2}$$

   (b) Show that $Y^{\perp}$ is a subspace of X' and $\dim(X/Y) = \dim Y^{\perp}$.

(c) Let $d \in \mathbb{N}$, and $\mathcal{P}_d$ denote polynomials with real coefficients of order at most d. Show that the functionals

$$\ell_j(p) = \partial_x^j p(0) \text{ for } j = 0, \ldots, d \tag{3}$$

are a basis for $\mathcal{P}'_d$. For $0 \leq d' < d$ use this basis to describe $\mathcal{P}_{d'}^{\perp}$.

4. Let X be a finite dimensional vector space over $\mathbb{C}$, and let $X_{\mathbb{R}}$ denote the vector space X, but with the scalar multiplication restricted to the real numbers. Prove that $\dim_{\mathbb{R}} X_{\mathbb{R}} = 2 \dim_{\mathbb{C}} X$. Show that $z \mapsto \bar{z}$ is a linear map from $\mathbb{C}_{\mathbb{R}} \to \mathbb{C}_{\mathbb{R}}$, but not from $\mathbb{C} \to \mathbb{C}$.

5. Suppose that $K \subset \mathbb{R}^2$ is a convex set. A point x lies on the boundary of K, bK, if, for any $\epsilon > 0$, $B_\epsilon(x) \cap K \neq \emptyset$, and $B_\epsilon(x) \cap K^c \neq \emptyset$. Show that if $x \in bK$, then there is a linear function $\ell_x : \mathbb{R}^2 \to \mathbb{R}$ so that

$$\ell_x(x) \geq \ell_x(y) \text{ for all } y \in K \setminus \{x\}. \tag{4}$$

When does the strict inequality hold for all $y \in K \setminus \{x\}$? The set $\{y : \ell_x(y) = \ell_x(x)\}$ is called a supporting line. Is the supporting line always unique?

6. Let $\ell : \mathbb{R}^2 \to \mathbb{R}$ be a linear function. A set of the form

$$H_{\ell,c} = \{x \in \mathbb{R}^2 : \ell(x) > c\} \tag{5}$$

is called an open half space. If $K \subset \mathbb{R}^2$ is a closed convex set, then show that

$$K = \bigcap_{H_{\ell,c} \supset K} H_{\ell,c}. \tag{6}$$

That is, K is the intersection of all the open half spaces that contain it. Prove that a closed unbounded, proper convex subset of $\mathbb{R}^2$ satisfies exactly one of the following criteria:

(a) K is a closed half space.

(b) K is the region between two parallel lines.

(c) K lies in a proper cone (the intersection of two half-spaces with non-parallel boundaries).

7. Let $X = \mathbb{R}^2$ and $Y = \{(x, 0) : x \in \mathbb{R}\}$, be a subspace. Suppose that we define a linear function ℓ on Y by setting $\ell((1, 0)) = 1$. For $1 \leq p < \infty$, define the norms

$$\|(x, y)\|_p = (x^p + y^p)^{\frac{1}{p}}, \tag{7}$$

and

$$\|(x, y)\|_\infty = \max\{|x|, |y|\}. \tag{8}$$

This linear function on Y satisfies

$$|\ell((x, 0))| \leq \|(x, 0)\|_p, \tag{9}$$

for all $1 \leq p \leq \infty$. We can linearly extend ℓ to all of $\mathbb{R}^2$ by setting

$$\ell((0, 1)) = \beta. \tag{10}$$

Denote this extension by ℓ_β. For each $1 \leq p \leq \infty$, find the values of β so that

$$|\ell_\beta((x, y))| \leq \|(x, y)\|_p, \text{ for all } (x, y) \in \mathbb{R}^2. \tag{11}$$

We can define another family of norms, for $0 < a < \infty$, by setting

$$N_a(x, y) = \sqrt{x^2 + a^2 y^2}. \tag{12}$$

For each $0 < a < \infty$, find the values of β so that

$$|\ell_\beta((x, y))| \leq N_a(x, y), \text{ for all } (x, y) \in \mathbb{R}^2. \tag{13}$$

8. Show for $0 < q < 1$, the function $d_q : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ defined by

$$d_q(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1}^{n} |x_i - y_i|^q \tag{14}$$

defines a metric on $\mathbb{R}^n$. How about $d_q(\boldsymbol{x}, \boldsymbol{y})^{\frac{1}{q}}$? What is

$$\lim_{q \to 0^+} d_q(\boldsymbol{x}, \boldsymbol{y})? \tag{15}$$

9. Let V be a vector space, possibly infinite dimensional.

   (a) Show that if $\mathscr{X} = \{x_\alpha : \alpha \in \mathscr{A}\} \subset V$ is a set of linearly independent vectors, then there is a basis for V of the form $\{x_\alpha : \alpha \in \mathscr{A}\} \cup \{y_\beta : \beta \in \mathscr{B}\}$. Hint: Let $\mathscr{W}$ consists of sets of linearly independent vectors in V, with the partial ordered defined by inclusion, then apply Zorn's lemma to prove this assertion.

   (b) Use this result to show that if $U \subset V$ is a subspace of V, then there exists another subspace W of V so that $V = U \oplus W$, and an isomorphism

$$\varphi : W \longrightarrow V/U. \tag{16}$$